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BASIC DYNAMICS OF POINT PARTICLES AND COLLECTIONS

Modern mechanics begins with the publication in 1687 of Isaac Newton's *Principia*, an extension of the work of his predecessors, notably Galileo and Descartes, that allows him to explain mathematically what he calls the "System of the World": the motions of planets, moons, comets, tides. The three "Axioms, or Laws of Motion" in the *Principia* (Newton, 1729) are:

> Law I: *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*
> Law II: *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*
> Law III: *To every Action there is always opposed an equal Reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

These axioms refer to the general behavior of a "body." It is clear from Newton's examples (projectiles, a top, planets, comets, a stone) in the same section that he intends these bodies to be macroscopic, ordinary objects.

But elsewhere Newton[1] refers to the "particles of bodies" in ways that suggest an atomic theory in which the primitive, elementary objects are small, indestructible, "solid, massy, hard, impenetrable, movable particles." These are what we will call the *point particles* of Newtonian physics. Newton says of them that, "these primitive Particles being Solids, are incomparably harder than any porous Bodies compounded of them; even so very hard as never to wear or break in pieces; no ordinary Power being able to divide what God himself made one in the first Creation."

The present chapter will begin with the assumption that Newton's three axioms refer fundamentally to these point particles. After deriving the laws of momentum, angular momentum, and work–energy for point particles, we will show that, given certain plausible and universally accepted additional axioms, essentially the same laws can be proved to apply to macroscopic bodies, considered as collections of the elementary point particles.

1.1 Newton's Space and Time

Before discussing the laws of motion of point masses, we must consider the space and time in which that motion takes place. For Newton, space was logically and physically distinct from the masses that might occupy it. Space provided a static, absolute, and independent reference with respect to which all particle positions and motions were

[1] See query 31, page 400 of Newton (1730).

to be measured. Space could be perceived by looking at the fixed stars which were presumed to be at rest relative to it. Newton also emphasized the ubiquity of space, comparing it to the *sensorium* of God.[2]

Newton thought of time geometrically, comparing it to a mathematical point moving steadily along a straight line. As with space, the even flow of time was absolute and independent of objects. He writes in the Principia, "Absolute, true and mathematical time, of itself, and from its own nature, flows equably without relation to anything external."[3]

In postulating an absolute space, Newton was breaking with Descartes, who held that the proper definition of motion was motion with respect to nearby objects. In the Principia, Newton uses the example of a spinning bucket filled with water to argue for absolute motion. If the bucket is suspended by a rope from a tree limb and then twisted, upon release the bucket will initially spin rapidly but the water will remain at rest. One observes that the surface of the water remains flat. Later, when the water has begun to rotate with the bucket, the surface of the water will now be concave, in response to the forces required to maintain its accelerated circular motion. If motion were to be measured with respect to proximate objects, one would expect the opposite observations. Initially, there is a large relative motion between the water and the proximate bucket, and later the two have nearly zero relative motion. So the Cartesian view would predict inertial effects initially, with the water surface becoming flat later, contrary to observation.

Newton realized that, as a practical matter, motion would often be measured by reference to objects rather than to absolute space directly. As we discuss in Section 16.1, the Galilean relativity principle states that Newton's laws hold when position is measured with respect to *inertial systems* that are either at rest, or moving with constant velocity, relative to absolute space. But Newton considered these relative standards to be secondary, merely stand-ins for space.

Nearly the opposite view was held by Newton's great opponent, Leibniz, who held that space is a "mere seeming thing" and that the only reality is the relation of objects. Their debate took the form of an exchange of letters, later published, between Leibniz and Clarke, Newton's surrogate.[4] Every student is urged to read them. The main difficulty for the modern reader is the abundance of theological arguments, mixed almost inextricably with the physical ones. One can appreciate the enormous progress that has been made since the seventeenth century in freeing physics from the constraints of theology. In the century after Newton and Leibniz, their two philosophical traditions continued to compete. But the success of the Newtonian method in explaining experiments and phenomena led to its gradual ascendency.[5]

[2]Seventeenth century physiology held that the information from human sense organs is collected in a "sensorium" which the soul then views.

[3]Newton's ideas about time were possibly influenced by those of his predecessor at Cambridge, Isaac Barrow. See Chapter 9 of Whitrow (1989).

[4]The correspondence is reprinted, with portions of Newton's writings, in Alexander (1956).

[5]Leibnizian ideas continued to be influential, however. The great eighteenth century mathematician Euler, to whom our subject owes so much, published in 1768 a widely read book, *Letters Addressed to a German Princess,* in which he explained the science of his day to the lay person (Euler, 1823). He felt it necessary to

Newton's space and time were challenged by Mach in the late nineteenth century. Mach argued, like Leibniz, that absolute space and time are illusory and that the only reality is the relation of objects.[6] Mach also proposed that the inertia of a particle is related to the existence of other particles and presumably would vanish without them, an idea that Einstein referred to as *Mach's Principle*.

Einstein's special relativity unifies space and time. And in his general relativity the metric of the combined *spacetime* becomes dynamic rather than static and absolute. General relativity is Machian in the sense that the masses of the universe affect the local *curvature* of spacetime, but Newtonian in the sense that spacetime itself (now represented by the dynamic metric field) is something all pervasive that has definite properties even at points containing no masses.

For the remainder of Part I of the book, we will adopt the traditional Newtonian definition of space and time. In Part II, we will consider the modifications of Lagrangian and Hamiltonian mechanics that are needed to accommodate special relativity, in which space and time are combined and time becomes a transformable coordinate.

1.2 Single Point Particle

In this section, we assume the applicability of Newton's laws to point particles, and introduce the basic derived quantities: momentum, angular momentum, work, kinetic energy, and their relations.

An uncharged point particle is characterized completely by its mass m and its position $\mathbf{r}$ relative to the origin of some inertial system of coordinates. The velocity $\mathbf{v} = d\mathbf{r}/dt$ and acceleration $\mathbf{a} = d\mathbf{v}/dt$ are derived by successive differentiation. Its momentum (which is what Newton called "motion" in his second law) is defined as

$$\mathbf{p} = m\mathbf{v} \tag{1.1}$$

Newton's second law then can be expressed as the law of momentum for point particles,

$$\mathbf{f} = \frac{d\mathbf{p}}{dt} \tag{1.2}$$

Since the mass of a point particle is unchanging, this is equivalent to the more familiar $\mathbf{f}=m\mathbf{a}$. The requirement that the change of momentum is "in the direction of the right line" of the impressed force $\mathbf{f}$ is guaranteed in modern notation by the use of vector quantities in the equations.

For the point particles, Newton's first law follows directly from eqn (1.2). When $\mathbf{f} = 0$, the time derivative of $\mathbf{p}$ is zero and so $\mathbf{p}$ is a constant vector. Note that eqn (1.2) is a *vector* relation. If, for example, the x-component of force f_x is zero, then the corresponding momentum component p_x will be constant regardless of what the other components may do.

devote some thirty pages of that book to refute Wolff, the chief proponent of Leibniz's philosophy. See also the detailed defense of Newton's ideas in Euler, L. (1748) "Reflexions sur l'Espace et le Tems [sic]" *Mémoires de l'Académie des Sciences de Berlin,* reprinted in Series III, Volume 2 of Euler (1911).

[6] See Mach (1907). Discussions of Mach's ideas are found in Rindler (1977, 2001) and Misner, Thorne and Wheeler (1973). A review of the history of spacetime theories from a Machian perspective is found in Barbour (1989, 2001). See also Barbour and Pfister (1995).

The angular momentum $\mathbf{j}$ of a point particle and the torque $\boldsymbol{\tau}$ acting on it are defined, respectively, as

$$\mathbf{j} = \mathbf{r} \times \mathbf{p} \qquad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{f} \tag{1.3}$$

It follows that the law of angular momentum for point particles is

$$\boldsymbol{\tau} = \frac{d\mathbf{j}}{dt} \tag{1.4}$$

since

$$\frac{d\mathbf{j}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{f} = 0 + \boldsymbol{\tau} \tag{1.5}$$

In a time dt the particle moves a vector distance $d\mathbf{r} = \mathbf{v}\, dt$. The work dW done by force $\mathbf{f}$ in this time is defined as

$$dW = \mathbf{f} \cdot d\mathbf{r} \tag{1.6}$$

This work is equal to the increment of the quantity $(1/2)mv^2$ since

$$dW = \mathbf{f} \cdot \mathbf{v}\, dt = \left(\frac{d(m\mathbf{v})}{dt}\, dt\right) \cdot \mathbf{v} = m\,(d\mathbf{v}) \cdot \mathbf{v} = d\left(\frac{1}{2}mv^2\right) \tag{1.7}$$

Taking a particle at rest to have zero kinetic energy, we define the kinetic energy T as

$$T = \frac{1}{2}mv^2 \tag{1.8}$$

with the result that a work–energy theorem for point particles may be expressed as $dW = dT$ or

$$\mathbf{f} \cdot \mathbf{v} = \frac{dT}{dt} \tag{1.9}$$

If the force $\mathbf{f}$ is either zero or constantly perpendicular to $\mathbf{v}$ (as is the case for purely magnetic forces on a charged particle, for example) then the left side of eqn (1.9) will vanish and the kinetic energy T will be constant.

1.3 Collective Variables

Now imagine a collection of N point particles labeled by index n, with masses m_1, $m_2, \ldots, m_N$ and positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$.

The other quantities defined in Section 1.2 will be indexed similarly, with $\mathbf{p}_n = m_n\mathbf{v}_n$, for example, referring to the momentum of the nth particle and $\mathbf{f}_n$ denoting the force acting on it. The total mass, momentum, force, angular momentum, torque, and kinetic energy of this collection may be defined by

$$M = \sum_{n=1}^{N} m_n \qquad \mathbf{P} = \sum_{n=1}^{N} \mathbf{p}_n \qquad \mathbf{F} = \sum_{n=1}^{N} \mathbf{f}_n \qquad \mathbf{J} = \sum_{n=1}^{N} \mathbf{j}_n \qquad \boldsymbol{\tau} = \sum_{n=1}^{N} \boldsymbol{\tau}_n \qquad T = \sum_{n=1}^{N} T_n \tag{1.10}$$

Note that, in the cases of $\mathbf{P}, \mathbf{F}, \mathbf{J}$, and $\boldsymbol{\tau}$, these are *vector* sums. If a particular collection consisted of two identical particles moving at equal speeds in opposite directions, for example, $\mathbf{P}$ would be zero.

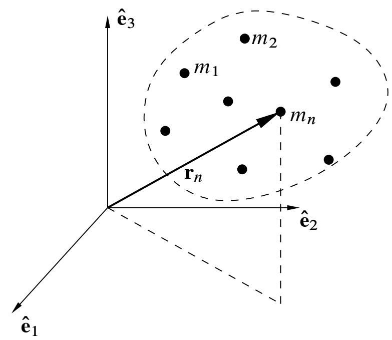

FIG. 1.1. A collection of point masses.

In the following sections, we derive the equations of motion for these collective variables. All of the equations of Section 1.2 are assumed to hold individually for each particle in the collection, with the obvious addition of subscripts n to each quantity to label the particular particle being considered. For example, $\mathbf{v}_n = d\mathbf{r}_n/dt$, $\mathbf{a}_n = d\mathbf{v}_n/dt$, $\mathbf{p}_n = m_n\mathbf{v}_n$, $\mathbf{f}_n = d\mathbf{p}_n/dt$, $\mathbf{f}_n = m_n\mathbf{a}_n$, etc.

1.4 The Law of Momentum for Collections

We begin with the law of momentum. Differentiation of the sum for $\mathbf{P}$ in eqn (1.10), using eqn (1.2) in the indexed form $d\mathbf{p}_n/dt = \mathbf{f}_n$, gives

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt}\sum_{n=1}^{N}\mathbf{p}_n = \sum_{n=1}^{N}\frac{d\mathbf{p}_n}{dt} = \sum_{n=1}^{N}\mathbf{f}_n = \mathbf{F} \tag{1.11}$$

The time rate of change of the total momentum is thus the total force.

But the force $\mathbf{f}_n$ on the nth particle may be examined in more detail. Suppose that it can be written as the vector sum of an external force $\mathbf{f}_n^{(\text{ext})}$ coming from influences operating on the collection from outside it, and an internal force $\mathbf{f}_n^{(\text{int})}$ consisting of all forces that cannot be identified as external, such as forces on particle n coming from collision or other interaction with other particles in the collection. For example, if the collection were a globular cluster of stars (idealized here as point particles!) orbiting a galactic center, the external force on star n would be the gravitational attraction from the galaxy, and the internal force would be the gravitational attraction of the other stars in the cluster. Thus

$$\mathbf{f}_n = \mathbf{f}_n^{(\text{ext})} + \mathbf{f}_n^{(\text{int})} \qquad \text{and, correspondingly,} \qquad \mathbf{F} = \mathbf{F}^{(\text{ext})} + \mathbf{F}^{(\text{int})} \tag{1.12}$$

where

$$\mathbf{F}^{(\text{ext})} = \sum_{n=1}^{N}\mathbf{f}_n^{(\text{ext})} \qquad \text{and} \qquad \mathbf{F}^{(\text{int})} = \sum_{n=1}^{N}\mathbf{f}_n^{(\text{int})} \tag{1.13}$$

Axiom 1.4.1: The Law of Momentum
It is taken as an axiom in all branches of modern physics that, insofar as the action of outside influences can be represented by forces, the following Law of Momentum must hold:

$$\mathbf{F}^{(\mathrm{ext})} = \frac{d\mathbf{P}}{dt} \tag{1.14}$$

It follows from this Law and eqn (1.11) that $\mathbf{F} = \mathbf{F}^{(\mathrm{ext})}$ and hence $\mathbf{F}^{(\mathrm{int})} = 0$. Identifying $\mathbf{P}$ with Newton's "motion" of a body, and $\mathbf{F}^{(\mathrm{ext})}$ with his "motive force impressed" on it, eqn (1.14) simply restates Newton's second law for bodies, now considered as collections of point particles.

An immediate consequence of the Law of Momentum is that the vanishing of $\mathbf{F}^{(\mathrm{ext})}$ makes $\mathbf{P}$ constant. We then say that $\mathbf{P}$ is *conserved*. This rule of momentum conservation is generally believed to apply even for those situations that cannot be described correctly by the concept of force. This is the essential content of Newton's first law. The total momentum of an isolated body does not change.

1.5 The Law of Angular Momentum for Collections

The derivation of the Law of Angular Momentum is similar to the previous Section 1.4. Differentiation of the sum for $\mathbf{J}$ in eqn (1.10), using eqn (1.4) in the indexed form $d\mathbf{j}_n/dt = \boldsymbol{\tau}_n$, gives

$$\frac{d\mathbf{J}}{dt} = \frac{d}{dt}\sum_{n=1}^{N}\mathbf{j}_n = \sum_{n=1}^{N}\frac{d\mathbf{j}_n}{dt} = \sum_{n=1}^{N}\boldsymbol{\tau}_n = \boldsymbol{\tau} \tag{1.15}$$

The time rate of change of the total angular momentum is thus the total torque.

Making the same division of forces into external and internal as was done in Section 1.4, we use the indexed form of eqn (1.3) to write the torque on particle n as the sum of external and internal torques,

$$\boldsymbol{\tau}_n = \mathbf{r}_n \times \mathbf{f}_n = \mathbf{r}_n \times \left(\mathbf{f}_n^{(\mathrm{ext})} + \mathbf{f}_n^{(\mathrm{int})}\right) = \boldsymbol{\tau}_n^{(\mathrm{ext})} + \boldsymbol{\tau}_n^{(\mathrm{int})} \tag{1.16}$$

where

$$\boldsymbol{\tau}_n^{(\mathrm{ext})} = \mathbf{r}_n \times \mathbf{f}_n^{(\mathrm{ext})} \qquad \text{and} \qquad \boldsymbol{\tau}_n^{(\mathrm{int})} = \mathbf{r}_n \times \mathbf{f}_n^{(\mathrm{int})} \tag{1.17}$$

Then, the total torque $\boldsymbol{\tau}$ defined in eqn (1.10) may then be written

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{(\mathrm{ext})} + \boldsymbol{\tau}^{(\mathrm{int})} \tag{1.18}$$

where

$$\boldsymbol{\tau}^{(\mathrm{ext})} = \sum_{n=1}^{N}\boldsymbol{\tau}_n^{(\mathrm{ext})} \qquad \text{and} \qquad \boldsymbol{\tau}^{(\mathrm{int})} = \sum_{n=1}^{N}\boldsymbol{\tau}_n^{(\mathrm{int})} \tag{1.19}$$

Axiom 1.5.1: The Law of Angular Momentum
It is taken as an axiom in all branches of modern physics that, insofar as the action of outside influences can be represented by forces, the following Law of Angular Momentum must hold:

$$\boldsymbol{\tau}^{(\mathrm{ext})} = \frac{d\mathbf{J}}{dt} \tag{1.20}$$

It follows from this Law and eqn (1.15) that $\boldsymbol{\tau} = \boldsymbol{\tau}^{(\text{ext})}$ and hence $\boldsymbol{\tau}^{(\text{int})} = 0$. An immediate consequence of the Law of Angular Momentum is that the vanishing of $\boldsymbol{\tau}^{(\text{ext})}$ makes $\mathbf{J}$ constant. We then say that $\mathbf{J}$ is *conserved.* This rule of angular momentum conservation is generally believed to apply even for those situations that cannot be described correctly by the concept of force. The total angular momentum of an isolated body does not change.

It is important to notice that the Laws of Momentum and Angular Momentum are *vector* relations. For example, in eqn (1.14), if $F_y^{(\text{ext})} = 0$ then P_y is conserved regardless of the values of the other components of the total external force. A similar separation of components holds also in eqn (1.20).

1.6 "Derivations" of the Axioms

Although the Law of Momentum is an axiom, it can actually be "derived" if one accepts an outdated action-at-a-distance model of internal forces in which the force $\mathbf{f}_n^{(\text{int})}$ is taken as the instantaneous vector sum of forces on particle n coming from all of the other particles in the collection. Denote the force on particle n coming from particle n' as $\mathbf{f}_{nn'}$ and thus write

$$\mathbf{f}_n^{(\text{int})} = \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \mathbf{f}_{nn'} \qquad \text{and hence} \qquad \mathbf{F}^{(\text{int})} = \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \mathbf{f}_{nn'} \tag{1.21}$$

In this model, Newton's third law applied to the point particles implies that

$$\mathbf{f}_{nn'} = -\mathbf{f}_{n'n} \tag{1.22}$$

which makes the symmetric double sum in eqn (1.21) vanish identically. With $\mathbf{F}^{(\text{int})} = 0$, eqns (1.11, 1.12) then imply eqn (1.14), as was to be proved. Equation (1.22) is sometimes referred to as the *weak form of Newton's third law.* We emphasize, however, that the Law of Momentum is more general than the action-at-a-distance model of the internal forces used in this derivation.

The Law of Angular Momentum is also an axiom but, just as in the case of linear momentum, it too can be "derived" from an outdated action-at-a-distance model of internal forces. We again denote the force on particle n coming from particle n' as $\mathbf{f}_{nn'}$ and thus write

$$\boldsymbol{\tau}_n^{(\text{int})} = \mathbf{r}_n \times \mathbf{f}_n^{(\text{int})} = \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \mathbf{r}_n \times \mathbf{f}_{nn'} \qquad \text{and hence} \qquad \boldsymbol{\tau}^{(\text{int})} = \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n'\neq n}}^{N} \mathbf{r}_n \times \mathbf{f}_{nn'} \tag{1.23}$$

It follows from eqn (1.22) that the second of eqn (1.23) may be rewritten as[7]

$$\boldsymbol{\tau}^{(\text{int})} = \frac{1}{2} \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n'\neq n}}^{N} (\mathbf{r}_n - \mathbf{r}_{n'}) \times \mathbf{f}_{nn'} \tag{1.24}$$

If we now assume (which we did *not* need to assume in the linear momentum case) that the force $\mathbf{f}_{nn'}$ is *central,* that is parallel (or anti-parallel) to the line $(\mathbf{r}_n - \mathbf{r}_{n'})$ between

[7] See Exercise 1.18 for a similar pattern.

particles n and n', then it follows from the vanishing of the cross products that $\boldsymbol{\tau}^{(\mathrm{int})}$ is zero, as was to be proved.

The addition of centrality to eqn (1.22) is sometimes called the *strong form of Newton's third law.* We emphasize that, as in the case of linear momentum, the Law of Angular Momentum is more general than the model of central, action-at-a-distance internal forces used in this last derivation.

For example, the laws of momentum and angular momentum can be applied correctly to the behavior of physical objects such as quartz spheres, whose internal structure requires modern solid-state physics for its description rather than Newtonian central forces between point masses. Yet, when there are identifiable external force fields acting, such as gravity for example, these objects will obey Axioms 1.4.1 and 1.5.1.

1.7 The Work–Energy Theorem for Collections

The work–energy theorem of eqn (1.9) can be extended to collections. Using the definition in eqn (1.10) together with the indexed form of eqn (1.8), the total kinetic energy can be written

$$T = \sum_{n=1}^{N} T_n = \frac{1}{2} \sum_{n=1}^{N} m_n v_n^2 \qquad \text{with} \qquad v_n^2 = \mathbf{v}_n \cdot \mathbf{v}_n \tag{1.25}$$

Then the time rate of change of T is equal to the rate at which work is done on all particles of the collection,

$$\frac{dT}{dt} = \sum_{n=1}^{N} \mathbf{f}_n \cdot \mathbf{v}_n \tag{1.26}$$

To prove this result, differentiate the sum for T in eqn (1.10), using eqn (1.9) in its indexed form $dT_n/dt = \mathbf{f}_n \cdot \mathbf{v}_n$ where $\mathbf{v}_n = d\mathbf{r}_n/dt$. Then

$$\frac{dT}{dt} = \frac{d}{dt} \sum_{n=1}^{N} T_n = \sum_{n=1}^{N} \frac{dT_n}{dt} = \sum_{n=1}^{N} \mathbf{f}_n \cdot \mathbf{v}_n \tag{1.27}$$

as was to be proved.

There is little benefit to introducing the separation of force $\mathbf{f}_n$ into external and internal terms here, since the total kinetic energy T can be changed even when no external forces are present. For example, consider four identical particles initially at rest at the four corners of a plane square. If there is a gravitational internal force among those particles, they will begin to collapse toward the center of the square. Thus T will increase even though only internal forces are acting.

1.8 Potential and Total Energy for Collections

In some cases, there will exist a potential function $U = U(\mathbf{r}_1, \ldots, \mathbf{r}_N, t)$ from which all forces on all particles can be derived. Thus

$$\mathbf{f}_n = -\nabla_n U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) = -\frac{\partial}{\partial \mathbf{r}_n} U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) \tag{1.28}$$

where[8]

$$\nabla_n = \frac{\partial}{\partial \mathbf{r}_n} = \sum_{i=1}^{3} \hat{\mathbf{e}}_i \frac{\partial}{\partial x_{ni}} \tag{1.29}$$

and x_{ni} is the i^{th} coordinate of the nth particle of the collection, that is, $\mathbf{r}_n = \sum_{i=1}^{3} x_{ni} \hat{\mathbf{e}}_i$.

The total energy E is defined as $E = T + U$, where T is the total kinetic energy. Its rate of change is

$$\frac{dE}{dt} = \frac{\partial U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)}{\partial t} \tag{1.30}$$

To see this, use the chain rule of partial differentiation and eqns (1.27, 1.28) to write

$$\frac{dT}{dt} = \sum_{n=1}^{N} \mathbf{f}_n \cdot \mathbf{v}_n = -\sum_{n=1}^{N} \mathbf{v}_n \cdot \frac{\partial U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)}{\partial \mathbf{r}_n} = -\left(\frac{dU}{dt} - \frac{\partial U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)}{\partial t}\right) \tag{1.31}$$

where the last equality implies eqn (1.30).

If the potential function $U = U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$ happens not to depend explicitly on the time t, the partial derivative in eqn (1.30) will vanish and E will be a constant. The total energy of the collection is then said to be *conserved.*

1.9 The Center of Mass

All of the collective variables in eqn (1.10) are simple scalar or vector sums of individual quantities. The center of mass of the collection $\mathbf{R}$ is only slightly more complicated. It is defined as the mass-weighted average position of the particles making up the collection,

$$\mathbf{R} = \frac{1}{M} \sum_{n=1}^{N} m_n \mathbf{r}_n \tag{1.32}$$

This $\mathbf{R}$ can be used to define a new set of position vectors $\boldsymbol{\rho}_n$ for the point particles, called *relative position vectors,* that give the positions of masses relative to the center of mass, rather than relative to the origin of coordinates as the $\mathbf{r}_n$ do.

The definition is

$$\boldsymbol{\rho}_n = \mathbf{r}_n - \mathbf{R} \qquad \text{or, equivalently,} \qquad \mathbf{r}_n = \mathbf{R} + \boldsymbol{\rho}_n \tag{1.33}$$

The vector $\boldsymbol{\rho}_n$ can be thought of as the position of particle n as seen by an observer standing at the center of mass. The vectors $\boldsymbol{\rho}_n$ can be expanded in terms of Cartesian unit vectors $\hat{\mathbf{e}}_i$ as

$$\boldsymbol{\rho}_n = \sum_{i=1}^{3} \rho_{ni} \hat{\mathbf{e}}_i \tag{1.34}$$

Component ρ_{ni} will be called the ith *relative coordinate* of particle n.

[8]See Section A.11 for a discussion of the notation $\partial U / \partial \mathbf{r}_n$, including cautions about its proper use.

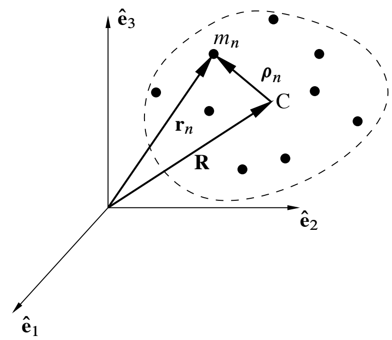

FIG. 1.2. Center of mass and relative position vectors. The center of mass is at C.

The velocity of the center of mass $\mathbf{V}$ is obtained by differentiating eqn (1.32) with respect to the time,

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{1}{M}\sum_{n=1}^{N} m_n \mathbf{v}_n \tag{1.35}$$

Then, differentiation of eqn (1.33) yields

$$\dot{\boldsymbol{\rho}}_n = \mathbf{v}_n - \mathbf{V} \qquad \text{or, equivalently,} \qquad \mathbf{v}_n = \mathbf{V} + \dot{\boldsymbol{\rho}}_n \tag{1.36}$$

where the definition $\dot{\boldsymbol{\rho}}_n = d\boldsymbol{\rho}_n/dt$ is used. This quantity will be called the *relative velocity* of mass m_n. It may be thought of as the apparent velocity of m_n as seen by an observer riding on the center of mass. A particular $\dot{\boldsymbol{\rho}}_n$ may in some cases be nonzero even when $\mathbf{v}_n = 0$ and the mass m_n is at rest relative to absolute space, due to the motion of the center of mass induced by motions of the other particles in the collection. Differentiating eqn (1.34) gives

$$\dot{\boldsymbol{\rho}}_n = \sum_{i=1}^{3} \dot{\rho}_{ni}\hat{\mathbf{e}}_i \tag{1.37}$$

where the $\dot{\rho}_{ni}$ will be called the ith *relative velocity coordinate* of mass m_n.

An observer standing at the center of mass will calculate the center of mass to be at his feet, at zero distance from him, as is shown in the following lemma which will be used in the later proofs.

Lemma 1.9.1: Properties of Relative Vectors
A very useful property of vectors $\boldsymbol{\rho}_n$ and $\dot{\boldsymbol{\rho}}_n$ is

$$0 = \sum_{n=1}^{N} m_n \boldsymbol{\rho}_n \qquad \text{and} \qquad 0 = \sum_{n=1}^{N} m_n \dot{\boldsymbol{\rho}}_n \tag{1.38}$$

Proof: The proof of the first expression follows directly from the definitions in eqns (1.32, 1.33),

$$\sum_{n=1}^{N} m_n \boldsymbol{\rho}_n = \sum_{n=1}^{N} m_n \left(\mathbf{r}_n - \mathbf{R}\right) = \sum_{n=1}^{N} m_n \mathbf{r}_n - \sum_{n=1}^{N} m_n \mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0 \tag{1.39}$$

with the second expression following from time differentiation of the first one. □

1.10 Center of Mass and Momentum

Having defined the center of mass, we now can write various collective quantities in terms of the vectors $\mathbf{R}, \boldsymbol{\rho}$ and their derivatives. The total momentum $\mathbf{P}$ introduced in eqn (1.10) can be expressed in terms of the total mass M and velocity of the center of mass $\mathbf{V}$ by the remarkably simple equation

$$\mathbf{P} = M\mathbf{V} \tag{1.40}$$

To demonstrate this result, we use the second of eqn (1.36) to rewrite $\mathbf{P}$ as

$$\mathbf{P} = \sum_{n=1}^{N} \mathbf{p}_n = \sum_{n=1}^{N} m_n \mathbf{v}_n = \sum_{n=1}^{N} m_n \left(\mathbf{V} + \dot{\boldsymbol{\rho}}_n\right) = \sum_{n=1}^{N} m_n \mathbf{V} + \sum_{n=1}^{N} m_n \dot{\boldsymbol{\rho}}_n = M\mathbf{V} \tag{1.41}$$

where the Lemma 1.9.1 was used to get the last equality. The total momentum of a collection of particles is the same as would be produced by a single particle of mass M moving with the center of mass velocity $\mathbf{V}$.

The Law of Momentum in eqn (1.14) can then be written, using eqn (1.40) and the constancy of M, as

$$\mathbf{F}^{(\text{ext})} = \frac{d\mathbf{P}}{dt} = M\mathbf{A} \qquad \text{where} \qquad \mathbf{A} = \frac{d\mathbf{V}}{dt} \tag{1.42}$$

is the acceleration of the center of mass. Thus, beginning from the assumption that $\mathbf{f} = m\mathbf{a}$ for individual point particles, we have demonstrated that $\mathbf{F}^{(\text{ext})} = M\mathbf{A}$ for composite bodies, provided that $\mathbf{A}$ is defined precisely as the acceleration of the center of mass of the body. This last result is very close to Newton's original second law.

1.11 Center of Mass and Angular Momentum

The total angular momentum $\mathbf{J}$ can also be rewritten in terms of center of mass and relative quantities. It is

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{1.43}$$

where

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} \qquad \text{and} \qquad \mathbf{S} = \sum_{n=1}^{N} \boldsymbol{\rho}_n \times \left(m_n \dot{\boldsymbol{\rho}}_n\right) \tag{1.44}$$

will be referred to as the "orbital" and "spin" contributions to $\mathbf{J}$, respectively. Note that $\mathbf{L}$ is just the angular momentum that would be produced by a single particle of mass M moving with the center of mass, and that $\mathbf{S}$ is just the apparent angular momentum that would be calculated by an observer standing on the center of mass and using only quantities relative to herself.

To demonstrate this result, we begin with eqn (1.10) and the indexed form of eqn (1.3) to write

$$\mathbf{J} = \sum_{n=1}^{N} \mathbf{j}_n = \sum_{n=1}^{N} \left(\mathbf{r}_n \times \mathbf{p}_n\right) = \sum_{n=1}^{N} \left(\mathbf{r}_n \times m_n \mathbf{v}_n\right) = \sum_{n=1}^{N} m_n \left(\mathbf{r}_n \times \mathbf{v}_n\right) \tag{1.45}$$

Now we introduce the definitions in eqns (1.33, 1.36), and use the linearity of cross products to get

$$\begin{aligned}\mathbf{J} &= \sum_{n=1}^{N} m_n \left(\mathbf{R} + \boldsymbol{\rho}_n\right) \times \left(\mathbf{V} + \dot{\boldsymbol{\rho}}_n\right) \\ &= \sum_{n=1}^{N} m_n \mathbf{R} \times \mathbf{V} + \sum_{n=1}^{N} m_n \mathbf{R} \times \dot{\boldsymbol{\rho}}_n + \sum_{n=1}^{N} m_n \boldsymbol{\rho}_n \times \mathbf{V} + \sum_{n=1}^{N} m_n \boldsymbol{\rho}_n \times \dot{\boldsymbol{\rho}}_n \\ &= \left\{\left(\sum_{n=1}^{N} m_n\right) \mathbf{R} \times \mathbf{V}\right\} + \left\{\mathbf{R} \times \left(\sum_{n=1}^{N} m_n \dot{\boldsymbol{\rho}}_n\right)\right\} \\ &\qquad + \left\{\left(\sum_{n=1}^{N} m_n \boldsymbol{\rho}_n\right) \times \mathbf{V}\right\} + \left\{\sum_{n=1}^{N} m_n \boldsymbol{\rho}_n \times \dot{\boldsymbol{\rho}}_n\right\}\end{aligned} \tag{1.46}$$

where, in each term in curly brackets, quantities not depending on index n have been factored out of the sum. Lemma 1.9.1 now shows that the second and third terms vanish identically. The remaining two terms are identical to the $\mathbf{L}$ and $\mathbf{S}$ defined in eqn (1.44), as was to be proved.

1.12 Center of Mass and Torque

The Law of Angular Momentum, eqn (1.20), contains the total external torque $\boldsymbol{\tau}^{(\text{ext})}$. Using eqns (1.17, 1.18), it may be written

$$\boldsymbol{\tau}^{(\text{ext})} = \sum_{n=1}^{N} \boldsymbol{\tau}_n^{(\text{ext})} = \sum_{n=1}^{N} \mathbf{r}_n \times \mathbf{f}_n^{(\text{ext})} \tag{1.47}$$

Substituting eqn (1.33) for $\mathbf{r}_n$ then gives

$$\boldsymbol{\tau}^{(\text{ext})} = \sum_{n=1}^{N} \left(\mathbf{R} + \boldsymbol{\rho}_n\right) \times \mathbf{f}_n^{(\text{ext})} = \mathbf{R} \times \sum_{n=1}^{N} \mathbf{f}_n^{(\text{ext})} + \sum_{n=1}^{N} \boldsymbol{\rho}_n \times \mathbf{f}_n^{(\text{ext})} = \boldsymbol{\tau}_{\text{o}}^{(\text{ext})} + \boldsymbol{\tau}_{\text{s}}^{(\text{ext})} \tag{1.48}$$

where we have defined the "orbital" and "spin" external torques as

$$\boldsymbol{\tau}_{\text{o}}^{(\text{ext})} = \mathbf{R} \times \mathbf{F}^{(\text{ext})} \qquad \text{and} \qquad \boldsymbol{\tau}_{\text{s}}^{(\text{ext})} = \sum_{n=1}^{N} \boldsymbol{\rho}_n \times \mathbf{f}_n^{(\text{ext})} \tag{1.49}$$

In a pattern that is becoming familiar, $\boldsymbol{\tau}_{\text{o}}^{(\text{ext})}$ is the torque that would result if the total external force on the collection acted on a particle at the center of mass, and $\boldsymbol{\tau}_{\text{s}}^{(\text{ext})}$ is the external torque on the collection that would be calculated by an observer standing at the center of mass and using $\boldsymbol{\rho}_n$ instead of $\mathbf{r}_n$ as the moment arm.

1.13 Change of Angular Momentum

The Law of Angular Momentum in eqn (1.20) may now be broken down into separate parts, one for the orbital angular momentum $\mathbf{L}$ and the other for the spin angular momentum **S**. The rate of change of **L** is equal to the orbital external torque,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_{\mathrm{o}}^{(\mathrm{ext})} \tag{1.50}$$

The demonstration is almost identical to that in Section 1.2 for the angular momentum of a single point particle,

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{R}\times\mathbf{P}) = \frac{d\mathbf{R}}{dt}\times\mathbf{P} + \mathbf{R}\times\frac{d\mathbf{P}}{dt} = \mathbf{V}\times M\mathbf{V} + \mathbf{R}\times\mathbf{F}^{(\mathrm{ext})} = 0 + \boldsymbol{\tau}_{\mathrm{o}}^{(\mathrm{ext})} \tag{1.51}$$

where eqns (1.40, 1.49) and the Law of Momentum, eqn (1.14), have been used. The rate of change of **S** is equal to the spin external torque,

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\tau}_{\mathrm{s}}^{(\mathrm{ext})} \tag{1.52}$$

The demonstration begins by using eqns (1.43, 1.48) to rewrite eqn (1.20) in the form

$$\boldsymbol{\tau}_{\mathrm{o}}^{(\mathrm{ext})} + \boldsymbol{\tau}_{\mathrm{s}}^{(\mathrm{ext})} = \frac{d\mathbf{L}}{dt} + \frac{d\mathbf{S}}{dt} \tag{1.53}$$

Equation (1.50) can then be used to cancel $d\mathbf{L}/dt$ with $\boldsymbol{\tau}_{\mathrm{o}}^{(\mathrm{ext})}$. Equating the remaining terms then gives eqn (1.52), as was to be shown.

Thus eqns (1.50, 1.52) give a separation of the Law of Angular Momentum into separate orbital and spin laws. The orbital angular momentum **L** and the orbital torque $\boldsymbol{\tau}_{\mathrm{o}}^{(\mathrm{ext})}$ are exactly what would be produced if all of the mass of the collection were concentrated into a point particle at the center of mass. The evolution of the orbital angular momentum defined by eqn (1.50) is totally independent of the fact that the collection may or may not be spinning about the center of mass.

Equation (1.52), on the other hand, shows that the time evolution of the spin angular momentum **S** is determined entirely by the external torque $\boldsymbol{\tau}_{\mathrm{s}}^{(\mathrm{ext})}$ measured by an observer standing at the center of mass, and is unaffected by the possible acceleration of the center of mass that may or may not be happening simultaneously.

1.14 Center of Mass and the Work–Energy Theorems

The total kinetic energy T may be expanded in the same way as the total angular momentum **J** in Section 1.13. We may use $T_n = m_n v_n^2/2$ and $v_n^2 = \mathbf{v}_n \cdot \mathbf{v}_n$ to rewrite eqn (1.10), and then use eqn (1.36) to get

$$T = \sum_{n=1}^{N} T_n = \frac{1}{2}\sum_{n=1}^{N} m_n \mathbf{v}_n \cdot \mathbf{v}_n = \frac{1}{2}\sum_{n=1}^{N} m_n \left(\mathbf{V} + \dot{\boldsymbol{\rho}}_n\right)\cdot\left(\mathbf{V} + \dot{\boldsymbol{\rho}}_n\right) \tag{1.54}$$

Expanding the dot product and using Lemma 1.9.1 then gives

$$T = T_{\mathrm{o}} + T_{\mathrm{I}} \tag{1.55}$$

where

$$T_{\mathrm{o}} = \frac{1}{2}MV^2 \qquad \text{and} \qquad T_{\mathrm{I}} = \frac{1}{2}\sum_{n=1}^{N} m_n \left\|\dot{\boldsymbol{\rho}}_n\right\|^2 \tag{1.56}$$

are the orbital and internal kinetic energies, respectively. The time rate of change of the orbital kinetic energy is

$$\frac{dT_o}{dt} = \mathbf{F}^{(ext)} \cdot \mathbf{V} \tag{1.57}$$

The demonstration uses eqn (1.40) and the Law of Momentum eqn (1.14),

$$\frac{dT_o}{dt} = \frac{d}{dt}\left(\frac{\mathbf{P}\cdot\mathbf{P}}{2M}\right) = \frac{\mathbf{P}}{M}\cdot\frac{d\mathbf{P}}{dt} = \mathbf{V}\cdot\mathbf{F}^{(ext)} \tag{1.58}$$

as was to be shown.

The time rate of change of the internal kinetic energy T_I is

$$\frac{dT_I}{dt} = \sum_{n=1}^{N} \mathbf{f}_n \cdot \dot{\boldsymbol{\rho}}_n \tag{1.59}$$

The demonstration of eqn (1.59) is quite similar to that of eqn (1.52). We begin with the collective work–energy theorem, eqn (1.27), rewritten using eqns (1.36, 1.55) as

$$\frac{dT_o}{dt} + \frac{dT_I}{dt} = \sum_{n=1}^{N} \mathbf{f}_n \cdot (\mathbf{V} + \dot{\boldsymbol{\rho}}_n) = \mathbf{V}\cdot\sum_{n=1}^{N} \mathbf{f}_n + \sum_{n=1}^{N} \mathbf{f}_n \cdot \dot{\boldsymbol{\rho}}_n \tag{1.60}$$

The earlier result in Section 1.4 that $\mathbf{F} = \mathbf{F}^{(ext)}$ then gives

$$\frac{dT_o}{dt} + \frac{dT_I}{dt} = \mathbf{V}\cdot\mathbf{F}^{(ext)} + \sum_{n=1}^{N} \mathbf{f}_n \cdot \dot{\boldsymbol{\rho}}_n \tag{1.61}$$

Using eqn (1.57) to cancel the first terms on each side gives eqn (1.59), as was to be shown. Note the *absence* of the superscript "(ext)" on $\mathbf{f}_n$ in eqn (1.59). This is not a mistake! The internal kinetic energy T_I can be changed by both external and internal forces, as we noted in Section 1.7.

1.15 Center of Mass as a Point Particle

It is remarkable that the center-of-mass motion of a body or other collection of point particles can be solved by imagining that the entire mass of the collection is a point particle at the center of mass $\mathbf{R}$ with the entire external force $\mathbf{F}^{(ext)}$ acting on that single point. The quantities and relations derived above,

$$\mathbf{P} = M\mathbf{V} \qquad \mathbf{F}^{(ext)} = \frac{d\mathbf{P}}{dt} \qquad \mathbf{L} = \mathbf{R}\times\mathbf{P} \qquad \frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}_o^{(ext)} = \mathbf{R}\times\mathbf{F}^{(ext)} \tag{1.62}$$

and

$$T_o = \frac{1}{2}MV^2 \qquad \frac{dT_o}{dt} = \mathbf{F}^{(ext)}\cdot\mathbf{V} \tag{1.63}$$

refer only to the total mass M, the center of mass $\mathbf{R}$, its derivative $\mathbf{V}$, and the total force $\mathbf{F}^{(ext)}$. And yet these formulas replicate all of the results obtained in Section 1.2 for a single point particle.

If, as we have assumed, Newton's laws apply fundamentally to Newtonian point particles, then these quantities and relations vindicate Newton's application of them to "bodies" rather than point particles. A billiard ball (by which we mean the center of a billiard ball) moves according to the same laws as a single point particle of the same mass.

1.16 Special Results for Rigid Bodies

The results obtained up to this point apply to all collections, whether they be solid bodies or a diffuse gas of point particles. Now we consider special, idealized collections called *rigid bodies.* They are defined by the condition that the distance $\|\mathbf{r}_n - \mathbf{r}_{n'}\|$ between any two masses in the collection is constrained to be constant. In Chapter 8 on the kinematics of rigid-body motion, we will prove that this constraint implies the existence of a (generally time-varying) vector $\boldsymbol{\omega}$ and the relation given in eqn (8.92),

$$\dot{\boldsymbol{\rho}}_n = \boldsymbol{\omega} \times \boldsymbol{\rho}_n \tag{1.64}$$

between each relative velocity vector and the corresponding relative location vector. This relation has a number of interesting applications which we will discuss in later chapters. Here we point out one of them, the effect on eqn (1.59). Rewriting that equation and using eqn (1.64) gives

$$\frac{dT_{\mathrm{I}}}{dt} = \sum_{n=1}^{N} \mathbf{f}_n \cdot \boldsymbol{\omega} \times \boldsymbol{\rho}_n = \boldsymbol{\omega} \cdot \sum_{n=1}^{N} \boldsymbol{\rho}_n \times \mathbf{f}_n = \boldsymbol{\omega} \cdot \sum_{n=1}^{N} (\mathbf{r}_n - \mathbf{R}) \times \mathbf{f}_n = \boldsymbol{\omega} \cdot (\boldsymbol{\tau} - \mathbf{R} \times \mathbf{F}) \tag{1.65}$$

where eqns (1.33, 1.10) have been used. But the Law of Momentum of Section 1.4 and the Law of Angular Momentum of Section 1.5 imply that

$$\mathbf{F} = \mathbf{F}^{(\mathrm{ext})} = \sum_{n=1}^{N} \mathbf{f}_n^{(\mathrm{ext})} \qquad \text{and} \qquad \boldsymbol{\tau} = \boldsymbol{\tau}^{(\mathrm{ext})} = \sum_{n=1}^{N} \mathbf{r}_n \times \mathbf{f}_n^{(\mathrm{ext})} \tag{1.66}$$

and hence that

$$\frac{dT_{\mathrm{I}}}{dt} = \boldsymbol{\omega} \cdot \left(\boldsymbol{\tau}^{(\mathrm{ext})} - \mathbf{R} \times \mathbf{F}^{(\mathrm{ext})} \right) \tag{1.67}$$

depends only on the external forces $\mathbf{f}_n^{(\mathrm{ext})}$. Thus, for rigid bodies and only for rigid bodies, we may add an "(ext)" to eqn (1.59) and write

$$\text{Rigid bodies only :} \qquad \frac{dT_{\mathrm{I}}}{dt} = \sum_{n=1}^{N} \mathbf{f}_n^{(\mathrm{ext})} \cdot \dot{\boldsymbol{\rho}}_n \tag{1.68}$$

It follows from eqns (1.55, 1.57, 1.68) that dT/dt for rigid bodies also depends only on external forces, and so we may write eqn (1.27) in the form

$$\text{Rigid bodies only :} \qquad \frac{dT}{dt} = \sum_{n=1}^{N} \mathbf{f}_n^{(\mathrm{ext})} \cdot \mathbf{v}_n. \tag{1.69}$$

1.17 Exercises

Exercise 1.1 In spherical polar coordinates, the radius vector is $\mathbf{r} = r\hat{\mathbf{r}}$.

(a) Use the product and chain rules of differentiation, and the partial derivatives read from eqns (A.48 – A.51), to obtain the standard expression for $\mathbf{v} = d\mathbf{r}/dt$ as

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}} \tag{1.70}$$

(b) By a similar process, derive the expression for $\mathbf{a} = d\mathbf{v}/dt$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}, \ddot{r}, \ddot{\theta}, \ddot{\phi}$.

Exercise 1.2 Derive the identities in eqns (A.76, A.78) and demonstrate that eqn (A.79) does follow from them.

Exercise 1.3 Consider a circular helix defined by

$$\mathbf{r} = a\,\cos\beta\,\hat{\mathbf{e}}_1 + a\,\sin\beta\,\hat{\mathbf{e}}_2 + c\beta\hat{\mathbf{e}}_3 \tag{1.71}$$

where a, c are given constants, and parameter β increases monotonically along the curve.

(a) Express the Serret–Frenet unit vectors $\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}$, the curvature ρ, and the torsion κ, in terms of $a, c, \beta, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$.

(b) Show that $\hat{\mathbf{n}}$ is always parallel to the x-y plane.

Exercise 1.4 In Section A.12 it is stated that the Serret–Frenet relations eqns (A.84, A.85, A.86) may be written as shown in eqn (A.87),

$$\frac{d\hat{\mathbf{t}}}{ds} = \boldsymbol{\omega} \times \hat{\mathbf{t}} \qquad \frac{d\hat{\mathbf{n}}}{ds} = \boldsymbol{\omega} \times \hat{\mathbf{n}} \qquad \frac{d\hat{\mathbf{b}}}{ds} = \boldsymbol{\omega} \times \hat{\mathbf{b}} \tag{1.72}$$

where $\boldsymbol{\omega} = \kappa\,\hat{\mathbf{t}} + \rho\,\hat{\mathbf{b}}$. Verify these formulas.

Exercise 1.5 A one tonne (1000 kg) spacecraft, in interstellar space far from large masses, explodes into three pieces. At the instant of the explosion, the spacecraft was at the origin of some inertial system of coordinates and had a velocity of 30 km/sec in the $+x$ direction relative to it. Precisely 10 sec after the explosion, two of the pieces are located simultaneously. They are a 300 kg piece at coordinates $(400, 50, -20)$ km and a 500 kg piece at coordinates $(240, 10, 32)$ km.

(a) Where was the third piece 10 sec after the explosion?

(b) Mission control wants to know where the missing piece will be 1 hour after the explosion. Give them a best estimate and an error circle. (Assume that the spacecraft had a largest dimension of 10 m, so that, at worst, a given piece might have come from a point 10 m from the center.)

(c) What if the spacecraft had been spinning end-over-end just before it exploded. Would the above answers change? At all? Appreciably? Explain.

Exercise 1.6 Three equal point masses $m_1 = m_2 = m_3 = m$ are attached to a rigid, massless rod of total length $2b$. Masses #1 and #3 are at the ends of the rod and #2 is in the middle. Mass m_1 is suspended from a frictionless pivot at the origin of an inertial coordinate system. Assume that the motion is constrained in a frictionless manner so that the masses all stay in the x-y plane. Let a uniform gravitational field $\mathbf{g} = g\hat{\mathbf{e}}_1$ act in the positive x-direction.

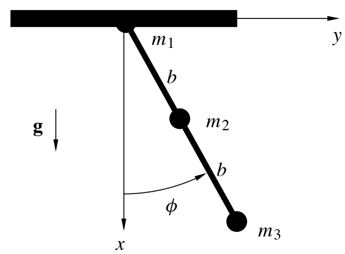

FIG. 1.3. Illustration for Exercise 1.6.

(a) Using plane polar coordinates, letting the r-direction be along the stick and letting ϕ be the angle between the stick and the x-axis, use the law of angular momentum to obtain $\ddot{\phi}$ and $\dot{\phi}^2$ as functions of ϕ.
(b) From the above, obtain $d^2\mathbf{r}_3/dt^2$ as a function of $\phi, \hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}$ and use

$$\mathbf{f}_3^{(\mathrm{int})} = m_3 \frac{d^2\mathbf{r}_3}{dt^2} - m_3\mathbf{g} \tag{1.73}$$

to obtain the internal force $\mathbf{f}_3^{(\mathrm{int})}$ on mass m_3.
(c) If it is entirely due to central forces from m_1 and m_2 as is required by the "strong form" of the second law, then $\mathbf{f}_3^{(\mathrm{int})}$ should be parallel to the stick. Is it? Explain.[9]

Exercise 1.7 Show clearly how eqns (1.55, 1.56) follow from eqn (1.54).

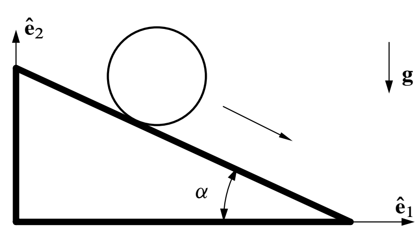

FIG. 1.4. Illustration for Exercise 1.8.

Exercise 1.8 A hollow, right-circular cylinder of mass M and radius a rolls without slipping straight down an inclined plane of angle α, starting from rest. Assume a uniform gravitational field $\mathbf{g} = -g\hat{\mathbf{e}}_2$ acting downwards.
(a) After the center of mass of the cylinder has fallen a distance h, what are the *vector* values of $\mathbf{V}, \mathbf{P}, \mathbf{S}$ for the cylinder? [Note: This question should be answered *without* considering the details of the forces acting. Assume that rolling without slipping conserves energy.]
(b) Using your results in part (a), find the force $\mathbf{F}^{(\mathrm{ext})}$ and spin torque $\boldsymbol{\tau}_s^{(\mathrm{ext})}$ acting on the cylinder.

Exercise 1.9 Write out eqn (A.74) and verify that it does express the correct chain rule result for df/dt.

[9] See Stadler, W. (1982) "Inadequacy of the Usual Newtonian Formulation for Certain Problems in Particle Mechanics," *Am. J. Phys.* **50**, p. 595.

Exercise 1.10 If all external forces $\mathbf{f}_n^{(\mathrm{ext})}$ on the point masses of a rigid body are derived from an *external* potential $U^{(\mathrm{ext})}(\mathbf{r}_1, \ldots, \mathbf{r}_D, t)$, show that the quantity $E = T + U^{(\mathrm{ext})}$ obeys

$$\frac{dE}{dt} = \frac{\partial U^{(\mathrm{ext})}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_D, t)}{\partial t} \tag{1.74}$$

Exercise 1.11 Let a collection of point masses $m_1, m_2, \ldots, m_N$ move without interaction in a uniform, external gravitational field $\mathbf{g}$ so that $\mathbf{f}_n = \mathbf{f}_n^{(\mathrm{ext})} = m_n \mathbf{g}$.

(a) Demonstrate that a possible potential for this field is

$$U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) = -\sum_{n=1}^{N} m_n \mathbf{r}_n \cdot \mathbf{g} \tag{1.75}$$

which may also be written as

$$U = -M\mathbf{R} \cdot \mathbf{g} \tag{1.76}$$

where M is the total mass of the collection, and $\mathbf{R}$ is its center of mass.

(b) Express $\mathbf{F}^{(\mathrm{ext})}$, $\boldsymbol{\tau}_{\mathrm{o}}^{(\mathrm{ext})}$, $\boldsymbol{\tau}_{\mathrm{s}}^{(\mathrm{ext})}$ in terms of $M, \mathbf{g}, \mathbf{R}$ for this collection.

(c) Which of the following are conserved: $E, \mathbf{P}, \mathbf{L}, \mathbf{S}, T_{\mathrm{o}}, T_{\mathrm{I}}$?

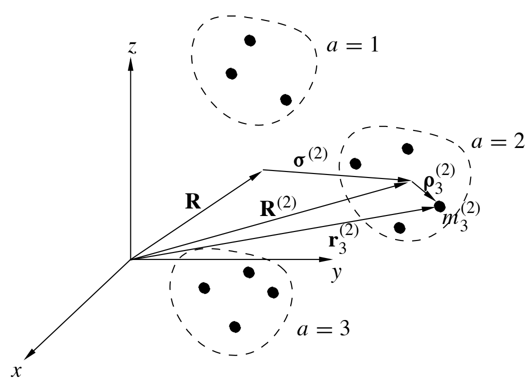

FIG. 1.5. Illustration for Exercise 1.12. Mass $m_3^{(2)}$ is the third mass in the second collection. Vector $\mathbf{R}^{(2)}$ is the center of mass of the second collection, and $\mathbf{R}$ is the center of mass of the entire system.

Exercise 1.12 Suppose that a total collection is made up of C sub-collections, labeled by the index $a = 1, \ldots, C$. The ath sub-collection has $N^{(a)}$ particles, mass $M^{(a)}$, momentum $\mathbf{P}^{(a)}$, center of mass $\mathbf{R}^{(a)}$, and center-of-mass velocity $\mathbf{V}^{(a)}$. (You might think of this as a globular cluster made up of stars. Each star is a sub-collection and the whole cluster is the total collection.)

(a) Demonstrate that the center of mass $\mathbf{R}$ and momentum $\mathbf{P}$ of the total collection may be written as

$$\mathbf{R} = \frac{1}{M}\sum_{a=1}^{C} M^{(a)}\mathbf{R}^{(a)} \qquad \mathbf{P} = \sum_{a=1}^{C} \mathbf{P}^{(a)} \tag{1.77}$$

where

$$M = \sum_{a=1}^{C} M^{(a)} \quad \text{and} \quad \mathbf{P}^{(a)} = M^{(a)}\mathbf{V}^{(a)} \tag{1.78}$$

i.e. that the total center of mass and total momentum may be calculated by treating each sub-collection as a single particle with all of its mass at its center of mass.
(b) Let the nth mass of the ath sub-collection $m_n^{(a)}$ have location $\mathbf{r}_n^{(a)}$. Define $\boldsymbol{\sigma}^{(a)} = \mathbf{R}^{(a)} - \mathbf{R}$ and $\boldsymbol{\rho}_n^{(a)} = \mathbf{r}_n^{(a)} - \mathbf{R}^{(a)}$ so that

$$\mathbf{r}_n^{(a)} = \mathbf{R} + \boldsymbol{\sigma}^{(a)} + \boldsymbol{\rho}_n^{(a)} \tag{1.79}$$

Prove the identities

$$\sum_{n=1}^{N^{(a)}} m_n^{(a)} \boldsymbol{\rho}_n^{(a)} = 0 \qquad \sum_{a=1}^{C} M^{(a)} \boldsymbol{\sigma}^{(a)} = 0 \tag{1.80}$$

and use them and their first time derivatives to demonstrate that the total angular momentum $\mathbf{J}$ may be written as

$$\mathbf{J} = \mathbf{L} + \mathbf{K} + \sum_{a=1}^{C} \mathbf{S}^{(a)} \tag{1.81}$$

where

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} \qquad \mathbf{K} = \sum_{a=1}^{C} \boldsymbol{\sigma}^{(a)} \times M^{(a)} \dot{\boldsymbol{\sigma}}^{(a)} \qquad \mathbf{S}^{(a)} = \sum_{n=1}^{N^{(a)}} \boldsymbol{\rho}_n^{(a)} \times m_n^{(a)} \dot{\boldsymbol{\rho}}_n^{(a)} \tag{1.82}$$

Note that $\mathbf{K}$ is just the spin angular momentum that would result if each sub-collection were a point mass located at its center of mass. Then the sum over $\mathbf{S}^{(a)}$ adds the intrinsic spins of the sub-collections.
(c) Suppose that a system consists of a massless stick of length b with six point masses, each of mass m, held rigidly by a massless frame at the vertices of a plane hexagon centered on one end of the stick. Similarly, four point masses, each also of mass m, are arranged at the vertices of a plane square centered on the other end. How far from the first end is the center of mass of the whole system? Do you need to assume that the hexagon and the square are coplanar?

Exercise 1.13 Consider a system consisting of two point masses, m_1 at vector location $\mathbf{r}_1$ and m_2 at $\mathbf{r}_2$, acted on only by action-at-a-distance internal forces $\mathbf{f}_{12}$ and $\mathbf{f}_{21}$, respectively. Denote the vector from the first to the second mass by $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, and its derivative by $d\mathbf{r}/dt = \mathbf{v}$. For this exercise, use the model in which the interaction between m_1 and m_2 is due entirely to these forces.

(a) Show that Axiom 1.4.1, implies that $\mathbf{f}_{21} + \mathbf{f}_{12} = 0$.
(b) Show that this and Axiom 1.5.1 imply that $\mathbf{f}_{21}$ and $\mathbf{f}_{12}$ must be parallel or anti-parallel to $\mathbf{r}$ (i.e., be central forces).
(c) Show that a potential of the form $U(\mathbf{r}_1, \mathbf{r}_2) = U_0 u(r)$, where U_0 is a constant and $u(r)$ is some function of $r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$, will produce forces $\mathbf{f}_{12} = -\partial U/\partial \mathbf{r}_1$ and $\mathbf{f}_{21} = -\partial U/\partial \mathbf{r}_2$ having the properties stated in parts (a) and (b) of this exercise.
(d) Show that $d\mathbf{V}/dt = 0$ and $\mu(d^2\mathbf{r}/dt^2) = \mathbf{f}_{21}$ where $\mathbf{V}$ is the velocity of the center of mass and $\mu = m_1 m_2/(m_1 + m_2)$. This μ is called the *reduced mass*.
(e) Write vectors $\mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\rho}_1$, and $\boldsymbol{\rho}_2$ as functions of $\mathbf{R}$ and $\mathbf{r}$.
(f) Write $\mathbf{P}, \mathbf{L}, \mathbf{S}, T_o$, and T_I in terms of μ, M, $\mathbf{R}, \mathbf{r}, \mathbf{V}$, and $\mathbf{v}$ only. Show that $\mathbf{S}$ is conserved.

Exercise 1.14 In part (c) of the previous Exercise, you showed that a potential of the form $U(\mathbf{r}_1, \mathbf{r}_2) = U_0 u(r)$ produces forces on the two masses that are equal, opposite, and central.

This problem demonstrates that such a potential can always be constructed. Assume that the two masses experience central forces $\mathbf{f}_{21} = f(r)\hat{\mathbf{r}} = -\mathbf{f}_{12}$ where $f(r)$ is some function of their separation $r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$. Define a potential function by $U(\mathbf{r}_1, \mathbf{r}_2) = -\int_{\mathbf{r}_{10}}^{\mathbf{r}_1} \mathbf{f}_{12}\cdot d\mathbf{r}_1 - \int_{\mathbf{r}_{20}}^{\mathbf{r}_2} \mathbf{f}_{21}\cdot d\mathbf{r}_2$.
(a) Demonstrate that $U(\mathbf{r}_1, \mathbf{r}_2)$ is independent of $\mathbf{R}$ and depends only on the scalars r and r_0, the initial and final separations of the two masses. Explain why this makes it independent of the integration path taken by the two particles, and hence single valued.
(b) Show that $\mathbf{f}_{12} = -\partial U/\partial \mathbf{r}_1$ and $\mathbf{f}_{21} = -\partial U/\partial \mathbf{r}_2$ recover the forces with which you began.

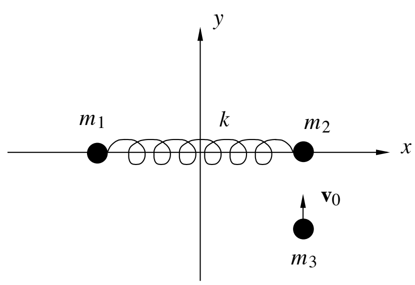

FIG. 1.6. Illustration for Exercise 1.15.

Exercise 1.15 Two masses m_1 and m_2 are connected by a massless spring of zero rest length, and force constant k. At time zero, the masses m_1 and m_2 lie at rest on the x axis at coordinates $(-a, 0, 0)$ and $(+a, 0, 0)$, respectively. Before time zero, a third mass m_3 is moving upwards with velocity $\mathbf{v}_0 = v_0\hat{\mathbf{e}}_2$, x-coordinate a, and y-coordinate less than zero. At time zero, m_3 collides with, and sticks to, m_2. Assume that the collision is impulsive, and is complete before m_1 or m_2 have changed position. Assume that the three masses are equal, with $m_1 = m_2 = m_3 = m$. Ignore gravity.
(a) Using the initial conditions of the problem to determine the constants of integration, write expressions for the center of mass vector $\mathbf{R}$ and the relative position vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ as functions of time for all $t > 0$.
(b) Write expressions for $\mathbf{r}_1$ and $\mathbf{r}_2$, the vector locations of masses m_1 and m_2, respectively, for all times $t > 0$.
(c) Show that mass m_1 has zero velocity at times $t_n = 2\pi n\sqrt{3m/2k}$, for $n = 0, 1, 2, \ldots$ but that the masses never return to the x axis.

Exercise 1.16 Prove that the $\mathbf{V}_\perp$ in eqn (A.4) can also be written as $\mathbf{V}_\perp = \hat{\mathbf{n}} \times (\mathbf{V} \times \hat{\mathbf{n}})$.

Exercise 1.17 Use eqn (A.68) to derive the related identity

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}\,(\mathbf{B}\cdot\mathbf{C}) \tag{1.83}$$

and show that the triple cross product is not associative.

Exercise 1.18 Refer to the rules in Section A.9 for this exercise.
(a) Verify the equality in eqn (A.54) by writing out both sides.
(b) Assume that $a_{ij} = -a_{ji}$. Verify each equality in the following formula

$$\sum_{i=1}^{3}\sum_{j=1}^{3} r_i a_{ij} = -\sum_{i=1}^{3}\sum_{j=1}^{3} r_i a_{ji} = -\sum_{j=1}^{3}\sum_{i=1}^{3} r_j a_{ij} = -\sum_{i=1}^{3}\sum_{j=1}^{3} r_j a_{ij} \tag{1.84}$$

(c) Use the results of part (b) to verify that $a_{ij} = -a_{ji}$ implies

$$\sum_{i=1}^{3}\sum_{j=1}^{3} r_i a_{ij} = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(r_i - r_j\right) a_{ij} \tag{1.85}$$

(d) Verify the equality in eqn (A.59).

Exercise 1.19 Refer to the rules in Section A.9 for this exercise.
(a) Demonstrate that

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij} x_i x_j = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(a_{ij} + a_{ji}\right) x_i x_j \tag{1.86}$$

(b) Use this result to verify eqn (A.63). In taking the partial derivatives, note that $\partial x_i / \partial x_k = \delta_{ik}$.
(c) Show that if $a_{ji} = -a_{ij}$ (anti-symmetric matrix) then the quadratic form $F(x)$ defined in eqn (A.62), and its partial derivatives, are zero.