Green's Reciprocity Theorem

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1 Derivation for Point Charges

Define a space-filling grid of point charges q_i. Points with no charge are represented as point charges with q_i equal to zero. Assume the nonzero ones are all at finite distance from the origin. Then the potential at point i is

$$v_i = \frac{1}{4\pi}\sum_{j=1}^{N}\frac{q_j}{r_{ij}} \tag{1}$$

where r_{ij} is the distance between points i and j.

Now define another collection of point charges on the same grid but with different values, denoted by using capital letters

$$V_i = \frac{1}{4\pi}\sum_{j=1}^{N}\frac{Q_j}{r_{ij}} \tag{2}$$

Form the two double sums

$$\sum_{i=1}^{N} q_i V_i = \frac{1}{4\pi}\sum_{i,j=1}^{N}\frac{q_i Q_j}{r_{ij}} \qquad \text{and} \qquad \sum_{i=1}^{N} Q_i v_i = \frac{1}{4\pi}\sum_{i,j=1}^{N}\frac{Q_i q_j}{r_{ij}} \tag{3}$$

Since $r_{ij} = r_{ji}$, and the sum is symmetric, it follows that

$$\sum_{i=1}^{N} q_i V_i = \sum_{i=1}^{N} Q_i v_i \tag{4}$$

This is the Greeen reciprocity theorem.[1]

2 Derivation for Continuous Distributions

With $dq = \rho_1(\mathbf{r})d\tau$ replacing q_i and $dQ = \rho_2(\mathbf{r})d\tau$ replacing Q_i, eq.(4) is equivalent to the continuous charge distribution expression

$$\int \rho_1(\mathbf{r})V_2(\mathbf{r})d\tau = \int \rho_2(\mathbf{r})V_1(\mathbf{r})d\tau \tag{5}$$

where V_1 corresponds to the v_i in eq.(1) and V_2 to the V_i in eq.(2). That is, the subscript 1 in eq.(5) corresponds to the lower-case letters in Section 1 and the subscript 2 in eq.(5) corresponds to the upper-case letters in Section 1. The index i is replaced by the spatial location vector $\mathbf{r}$, and it is still assumed that the charge densities are nonzero only for finite distance from the origin.[2]

3 Experiments Using Two Conductors

Consider two initially uncharged conductors A and B of any shape and any distance apart (except not touching, of course).

First do an experiment corresponding to the charge distribution and potential denoted by subscript 1 in eq.(5). In this first experiment, charge $q \neq 0$ is added to B, but A is left uncharged (*i.e.,* the integral of ρ_1 over equipotential surface A would produce a net charge of zero, but the integral of ρ_1 over the equipotential surface B would produce a net charge q.)

Second, without moving the conductors A and B or changing their orientations, do a different experiment corresponding to the charge distribution and potential denoted by subscript 2 in eq.(5). Again begin with both conductors uncharged. In this second experiment, charge $q \neq 0$ is added

[1] See Section 3-2 of Panofsky and Phillips [3]

[2] See Problem 3.50 of Griffiths [1] and Problem 1.12 of Jackson [2].

to A, but B is left uncharged (*i.e.,* the integral of ρ_2 over equipotential surface A would produce a net charge of q, but the integral of ρ_2 over the equipotential surface B would produce a net charge zero.)

The integral on the left side of eq.(5) then yields

$$\int \rho_1(\mathbf{r})V_2(\mathbf{r})d\tau = 0V_{A2} + qV_{B2} = qV_{B2} \tag{6}$$

and the integral on the right side of eq.(5) yields

$$\int \rho_2(\mathbf{r})V_1(\mathbf{r})d\tau = qV_{A1} + 0V_{B1} = qV_{A1} \tag{7}$$

From the Green reciprocity theorem eq.(5), the two integrals in eqn.(6, 7) are equal. Since q is assumed to be the same in the two experiments, the result is

$$V_{A1} = V_{B2} \tag{8}$$

The potential on uncharged conductor A when charge q is added to conductor B (as in experiment 1) is the same as the potential on uncharged condutor B when charge q is added to conductor A (as in experiment 2).

References

[1] D. J. Griffiths. *Introduction to Electrodynamics*. Pearson Education Ltd., 4th edition, 2013.

[2] J. D. Jackson. *Classical Electrodynamics*. John Wiley and Sons, New York, 2nd edition, 1975.

[3] W. K. Panofsky and M. Phillips. *Classical Electricity and Magnetism*. Addison-Wesley Pub. Co., 1955.