

Helmholtz Theorem and Uniqueness

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Abstract

Vector calculus in E^3 , three dimensions with a Euclidian metric, is the *lingua franca* of classical physics, including classical electrodynamics. This article corrects some long-standing imprecisions in a fundamental result.

Some textbooks assert that a vector function defined in the whole of a three dimensional space is uniquely determined by its divergence, its curl, and the condition that the function goes to zero as the radius (distance from an origin) goes to infinity. This article suggests that this condition is not sufficient for uniqueness. A proof is given that a sufficient condition for uniqueness is for the vector function to approach zero more rapidly than radius to the minus 3/2 power as the radius goes to infinity.

The issue is important because the same uniqueness condition also determines the uniqueness of the decomposition of a vector field into a transverse field plus a solenoidal field, as is done in the Coulomb gauge of electrodynamics.

1 Introduction

With the exception of the Dirac delta function in Appendix A, all functions in this article are assumed to have continuous derivatives to all orders. The statement that a field function $f(\mathbf{r})$ obeys $f = O(r^{-\alpha})$ as $r \rightarrow \infty$, where α is a given positive constant, is defined to mean that there exist a positive constant c and a radius r_0 such that $|f(\mathbf{r})| < cr^{-\alpha}$ for all $r > r_0$. (Equivalently, that $r^\alpha |f(\mathbf{r})|$ is bounded as $r \rightarrow \infty$.) For vector fields $\mathbf{F}(\mathbf{r})$ the statement that $\mathbf{F} = O(r^{-\alpha})$ is defined to mean that, for each cartesian component, $F_i = O(r^{-\alpha})$ where $i = [1, 2, 3]$.

Some standard electromagnetism textbooks assert that a vector function $\mathbf{F}(\mathbf{r})$ defined over the whole space E^3 is determined uniquely by its divergence, its curl, and the condition that \mathbf{F} vanishes at infinity. If that condition is interpreted as $\mathbf{F} = O(r^{-\alpha})$ for any positive constant α , then this article proves that condition to be too weak to imply uniqueness. The sufficient condition for uniqueness is proved here to be $\mathbf{F} = O(r^{-(3/2+\beta)})$, where β is any (possibly small) positive

constant. That is, α must be greater than $3/2$; function \mathbf{F} must go to zero more rapidly than $r^{-3/2}$.

Some textbook treatments of the uniqueness of a function given its divergence and curl in the whole of E^3 are discussed in Section 11. Two older textbooks give a condition for uniqueness that is sufficient but unnecessarily strong, and two more recent textbooks assert a condition that is too weak and not sufficient.

2 Basic Theorems

Divergence Theorem: Let $\mathbf{F}(x, y, z)$ be defined in a volume \mathcal{V} surrounded by a sufficiently smooth boundary \mathcal{S} . Denote the three dimensional volume element by $d\tau = d^3r$ and the outwards pointing surface element by $d\mathbf{a}$. Then the divergence ($\nabla \cdot \mathbf{F}$) is related to the surface integral by

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{F}) d\tau = \oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{a} \quad (2.1)$$

Stokes Theorem: Let $\mathbf{F}(x, y, z)$ be defined on a surface \mathcal{S} surrounded by a sufficiently smooth boundary line \mathcal{C} . (This surface need not be planar.) Denote the line element of the boundary by $d\ell$ and the surface element of \mathcal{S} by $d\mathbf{a}$. Assume $d\ell$ to point in the direction of the fingers of the right hand when its thumb points in the direction of $d\mathbf{a}$. Then the curl ($\nabla \times \mathbf{F}$) is related to the line integral by

$$\int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint_{\mathcal{C}} \mathbf{F} \cdot d\ell \quad (2.2)$$

3 Existence of Scalar Potential ϕ

Theorem 3.1: These three statements are equivalent. Any one of the following three items is true if and only if all three are true.

1. $\nabla \times \mathbf{F} = 0$ in the whole of three-dimensional space E^3 .
2. $\oint \mathbf{F} \cdot d\ell = 0$ for any closed line integral of \mathbf{F} in the three dimensional space.
3. There exists a single-valued function $\phi(\mathbf{r})$ such that $\mathbf{F} = -\nabla\phi$ at all points of the three-dimensional space.

Proof: Since the path of the closed line integral is arbitrary, the equivalence of Items 1 and 2 follows directly from Stokes Theorem.

To prove the equivalence of Item 3, choose a path from the origin to point x, y, z and define ϕ as

$$\phi(\mathbf{r}) = - \int_0^{\mathbf{r}} \mathbf{F} \cdot d\ell = - \int_0^{x,y,z} (F_x dx + F_y dy + F_z dz) \quad (3.1)$$

Then

$$\frac{\partial \phi}{\partial x} = -\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left(\int_0^{x+\delta, y, z} - \int_0^{x, y, z} \right) (F_x dx + F_y dy + F_z dz) = -F_x \quad (3.2)$$

with similar results for y and z , and hence

$$\mathbf{F}(\mathbf{r}) = -\nabla \phi(\mathbf{r}) \quad (3.3)$$

To prove ϕ single valued, note that any two *different* paths from the origin to x, y, z can be combined to form a closed path. This combined path will go from the origin to \mathbf{r} on one path and then, in the reverse sense, from \mathbf{r} back to the origin on the other path. But integration in a reverse sense on a path simply changes the sign of that integral. From item 2 above, the combined path must yield zero, and hence those two different paths from the origin to x, y, z must yield the same integral. The definition of ϕ in eqn (3.1) is path independent and ϕ is a well-defined, single-valued function.

Thus Item 2 implies Item 3. And, from the curl of eqn (3.3), Item 3 implies Item 1. But Items 1 and 2 are equivalent. Thus all three statements are equivalent. ■

4 Alternate Definition of Scalar Potential

Although the definition of ϕ in eqn (3.1) is adequate, it is not unique. If $\nabla \phi = -\mathbf{F}$ then it is also true that $\nabla(\phi + b) = -\mathbf{F}$, where b is any constant. It is useful to choose b to cancel the lower limit in eqn (3.1) and define an alternate single valued function $\tilde{\phi}$ as

$$\tilde{\phi}(\mathbf{r}) = \phi(\mathbf{r}) + b = - \int^{\mathbf{r}} \mathbf{F} \cdot d\ell \quad (4.1)$$

Of course, since

$$-\nabla \tilde{\phi} = -\nabla(\phi + b) = -\nabla \phi = \mathbf{F} \quad (4.2)$$

the alternate function still satisfies eqn (3.3).

If $\alpha > 1$, it follows from eqn (4.1) that the behaviors of \mathbf{F} and $\tilde{\phi}$ as $r \rightarrow \infty$ are related by

$$\mathbf{F} = O(r^{-\alpha}) \iff \tilde{\phi} = O(r^{-\alpha+1}) \quad (4.3)$$

5 Uniqueness of a Vector Field in a Finite Volume

A function $\mathbf{F}(\mathbf{r})$ is defined in a finite volume \mathcal{V} with surface S . From this function we can calculate its divergence $s(\mathbf{r})$ and curl $\mathbf{C}(\mathbf{r})$

$$\nabla \cdot \mathbf{F} = s \quad \nabla \times \mathbf{F} = \mathbf{C} \quad (5.1)$$

Denoting the outward pointing surface element of S as $d\mathbf{a} = \hat{\mathbf{n}} da$, we can also calculate the outward normal

$$F_n = \hat{\mathbf{n}} \cdot \mathbf{F} \quad (5.2)$$

Theorem 5.1: A function \mathbf{F} defined in a finite volume \mathcal{V} with surface S is uniquely determined by its divergence and curl in \mathcal{V} , and its outward normal on S , in the sense that any other function that has these same values must be equal to \mathbf{F} .

Proof: Suppose two functions \mathbf{F}^* and \mathbf{F} both to have the divergence, curl, and outward normal given in eqn (5.1) and eqn (5.2). The proof is to define

$$\mathbf{W} = \mathbf{F}^* - \mathbf{F} \quad (5.3)$$

and then prove $\mathbf{W} = 0$.

This definition of \mathbf{W} implies that

$$\nabla \cdot \mathbf{W} = 0 \quad \nabla \times \mathbf{W} = 0 \quad W_n = 0 \quad (5.4)$$

Since $\nabla \times \mathbf{W} = 0$ it follows from Section 3 with \mathbf{W} in place of \mathbf{F} that there exists a scalar function ϕ such that

$$\mathbf{W} = -\nabla\phi \quad (5.5)$$

Since $\nabla \cdot \mathbf{W} = 0$, this ϕ obeys

$$\nabla^2\phi = -\nabla \cdot \mathbf{W} = 0 \quad (5.6)$$

Expanding the divergence of $(\phi\nabla\phi)$ and using eqn (5.5) and eqn (5.6) gives

$$\nabla \cdot (\phi\nabla\phi) = \phi(\nabla^2\phi) + (\nabla\phi) \cdot (\nabla\phi) = (\nabla\phi) \cdot (\nabla\phi) \quad (5.7)$$

and hence

$$\nabla \cdot (\phi\nabla\phi) = (\nabla\phi) \cdot (\nabla\phi) = W^2 \quad (5.8)$$

Now apply the divergence theorem to give

$$\int_{\mathcal{V}} W^2 d\tau = \int_{\mathcal{V}} \nabla \cdot (\phi\nabla\phi) d\tau = \oint_S (\phi\nabla\phi) \cdot d\mathbf{a} = - \oint_S (\phi W_n da) \quad (5.9)$$

Since $W_n = (F_n^* - F_n) = 0$, and W^2 is positive definite, it follows that $\mathbf{W} = 0$ and hence $\mathbf{F}^* = \mathbf{F}$, which proves uniqueness in \mathcal{V} . ■

Caution: It is tempting to apply Theorem 5.1 by assuming condition $\mathbf{F} \rightarrow 0$ as $r \rightarrow \infty$ to mean that $W_n = 0$ on a surface S at *infinity*, which would prove uniqueness. But there is no such thing as a surface at *infinity*. Infinity is a limit, not a location. When the volume \mathcal{V} expands to include the whole of E^3 , proof of uniqueness requires a limiting process as surface S goes to *infinity*, which it never reaches. A correct proof of uniqueness in the whole of E^3 is given in Section 6.

6 Uniqueness of a Vector Field in the Whole of E^3

If \mathcal{V} is extended to be the whole space, the uniqueness condition $W_n = 0$ in eqn (5.9) can be replaced by the condition that \mathbf{W} goes to zero sufficiently rapidly as $r \rightarrow \infty$.

Theorem 6.1: A function \mathbf{F} defined on all space is uniquely determined by its divergence, its curl, and the condition that, for some (possibly small) positive constant β , $\mathbf{F} = O(r^{-(3/2+\beta)})$ as $r \rightarrow \infty$, in the sense that any other function that meets these same conditions must be equal to \mathbf{F} .

Proof: As in Section 5, assume that functions \mathbf{F}^* and \mathbf{F} have the same divergence and curl and define $\mathbf{W} = (\mathbf{F}^* - \mathbf{F})$. We prove that $\mathbf{W} = 0$, which implies $\mathbf{F}^* = \mathbf{F}$ and hence uniqueness.

It follows from the definition of \mathbf{W} that $\nabla \cdot \mathbf{W} = 0$ and $\nabla \times \mathbf{W} = 0$. Theorem 3.1 with \mathbf{W} in place of \mathbf{F} then permits the definition of a scalar function ϕ with $\mathbf{W} = -\nabla\phi$. The eqn (4.1) of Section 4 can then be used to define an alternate function $\tilde{\phi}$, such that

$$\mathbf{W} = -\nabla\tilde{\phi} \quad \text{and hence} \quad \nabla^2\tilde{\phi} = \nabla \cdot \mathbf{W} = 0 \quad (6.1)$$

Expanding the divergence of $(\tilde{\phi} \nabla\tilde{\phi})$ and using eqn (6.1) gives

$$\nabla \cdot (\tilde{\phi} \nabla\tilde{\phi}) = \tilde{\phi} (\nabla^2\tilde{\phi}) + (\nabla\tilde{\phi}) \cdot (\nabla\tilde{\phi}) = (\nabla\tilde{\phi}) \cdot (\nabla\tilde{\phi}) \quad (6.2)$$

and hence

$$\nabla \cdot (\tilde{\phi} \nabla\tilde{\phi}) = (\nabla\tilde{\phi}) \cdot (\nabla\tilde{\phi}) = W^2 \quad (6.3)$$

Applying the divergence theorem for a volume \mathcal{V}_ρ enclosed by a spherical surface S_ρ of radius ρ centered on the origin gives

$$\int_{\mathcal{V}_\rho} W^2 d\tau = \int_{\mathcal{V}_\rho} \nabla \cdot (\tilde{\phi} \nabla\tilde{\phi}) d\tau = \oint_{S_\rho} (\tilde{\phi} \nabla\tilde{\phi}) \cdot d\mathbf{a} \quad (6.4)$$

As $\rho \rightarrow \infty$, this \mathcal{V}_ρ will expand to become the whole space.

Suppose that $\mathbf{W} = -\nabla\tilde{\phi} = O(r^{-\alpha})$ as $r \rightarrow \infty$, for some $\alpha > 1$. From eqn (4.3) it follows that $\tilde{\phi} = O(r^{-\alpha+1})$ and hence $(\tilde{\phi} \nabla\tilde{\phi}) = O(r^{-2\alpha+1})$. Taking account of the fact that $da = r^2 d\Omega$ for an increment of solid angle $d\Omega$, the surface integral in eqn (6.4) is then $O(r^{-2\alpha+3})$ and will approach zero as $r \rightarrow \infty$ provided that $-2\alpha + 3 < 0$ and hence $\alpha > 3/2$. If the two functions \mathbf{F}^* and \mathbf{F} , and thus their difference \mathbf{W} , obey $O(r^{-(3/2+\beta)})$, for some positive constant β , the surface integral in eqn (6.4) vanishes as $\rho \rightarrow \infty$ and \mathcal{V}_ρ becomes the whole of E^3 . Since W^2 is positive definite, it then follows from eqn (6.4) that $\mathbf{W} = 0$ and hence $\mathbf{F}^* = \mathbf{F}$, which proves uniqueness. The condition for uniqueness can also be stated as $\tilde{\phi} = O(r^{-(1/2+\beta)})$. (As a check, notice that $\alpha = 3/2 + \beta > 1$ as was assumed.) ■

Note 6.1: It follows from Theorem 6.1 that any vector function which has zero divergence, zero curl, and goes to zero faster than $r^{-3/2}$ as $r \rightarrow \infty$ must be identically zero. There is no (nonzero) function that has zero divergence, zero curl, and goes to zero faster than $r^{-3/2}$ as $r \rightarrow \infty$.

7 Solution of the Poisson Equation in E^3

Theorem 7.1: Given a source function $f(\mathbf{r})$, the Poisson equation for function $U(\mathbf{r})$ is (with the conventional minus sign)

$$\nabla^2 U(\mathbf{r}) = -f(\mathbf{r}) \quad (7.1)$$

If $f = O(r^{-(2+\varepsilon)})$ as $r \rightarrow \infty$, where ε is some positive constant, then a solution of the Poisson Equation at point \mathbf{r}_0 is

$$U(\mathbf{r}_0) = \frac{1}{4\pi} \int \frac{f(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} d\tau \quad (7.2)$$

Proof: Spherical polar coordinates can be introduced in eqn (7.2) without loss of generality. Then $d\tau = r^2 d\Omega dr$ and the integrand in eqn (7.2) is $O(r^{-(2+\varepsilon)} r^{-1} r^2)$ which is $O(r^{-(1+\varepsilon)})$, and hence the integral defining $U(\mathbf{r}_0)$ converges.

As shown in Appendix A, the condition $f = O(r^{-(2+\varepsilon)})$ is also sufficient for the use of Green's identity to establish the correctness of eqn (7.2). ■

Note 7.1: When, as here, the Poisson Equation is assumed to hold in the whole of space E^3 , with only the condition $f = O(r^{-(2+\varepsilon)})$ to ensure the convergence of eqn (7.2), then its solution $U(\mathbf{r})$ is not uniquely determined. For if U is a solution then $\tilde{U} = (U + b)$ is also a solution, where b is any constant.

However, even though U itself is not uniquely determined, its gradient may be. If ∇U obeys the condition $\nabla U = O(r^{-(3/2+\beta)})$, then Theorem 6.1 proves that ∇U is uniquely determined by its divergence $\nabla \cdot (\nabla U) = -f$ and curl $\nabla \times \nabla U = 0$.

Note 7.2: If f is nonzero only at finite distance from the origin (a more stringent condition than $f = O(r^{-(2+\varepsilon)})$), then eqn (7.2) implies that $\nabla U = O(r^{-2})$. Since $2 > 3/2$, the vector ∇U will then be uniquely determined. For example, putting f and $-\nabla U$ equal to charge density and electric field, respectively, the electric field of a static, spatially finite charge distribution is uniquely determined.

8 Unique Solution of Poisson Equation in Finite Volumes

Computation of U is often aided by uniqueness theorems involving boundary conditions, specification of U on the boundary surface S of a volume \mathcal{V} .

Let U^* and U be any two potentially different solutions to eqn (7.1) and define $\eta = (U^* - U)$. Any boundary condition sufficient to make $\eta = 0$ in the whole of \mathcal{V} will imply uniqueness, in the sense that there can be only one solution in \mathcal{V} satisfying that boundary condition.

Since we have both eqn (7.1) and $\nabla^2 U^*(\mathbf{r}) = -f(\mathbf{r})$ it follows that $\nabla^2 \eta = 0$. Following the pattern in Section 5 gives

$$\nabla \cdot (\eta \nabla \eta) = \eta (\nabla^2 \eta) + (\nabla \eta) \cdot (\nabla \eta) = (\nabla \eta) \cdot (\nabla \eta) \quad (8.1)$$

$$\int_{\mathcal{V}} |\nabla\eta|^2 d\tau = \int_{\mathcal{V}} (\nabla\eta) \cdot (\nabla\eta) d\tau = \int_{\mathcal{V}} \nabla \cdot (\eta\nabla\eta) = \int_{\mathcal{S}} \eta\nabla\eta \cdot da \quad (8.2)$$

We now use eqn (8.2) to discuss two classes of boundary conditions, Dirichlet and Neumann.

1. **Dirichlet Boundary Condition:** Suppose U to be specified on \mathcal{S} so that $U^* = U$ there. Then $\eta = 0$ on \mathcal{S} , and the vanishing of the surface integral in eqn (8.2) shows that the positive definite quantity $|\nabla\eta|$ is zero in the whole of \mathcal{V} . The vanishing of gradient $\nabla\eta = 0$ everywhere in \mathcal{V} , together with $\eta = 0$ on \mathcal{S} , imply $\eta = 0$ for the whole of \mathcal{V} . Thus the two possibly different solutions U^* and U are equal. There is only one unique solution in \mathcal{V} that has the specified boundary value on \mathcal{S} .

For the particular example of the Laplace equation (eqn (7.1) with $f = 0$), an obvious possible solution is $U = 0$. A Dirichlet boundary condition that $U = 0$ on \mathcal{S} proves that solution to be unique. For the Laplace equation, the only solution that vanishes on \mathcal{S} is the solution that also vanishes in the whole of \mathcal{V} .

2. **Neumann Boundary Condition:** Suppose ∇U to be given on \mathcal{S} so that $\nabla U^* = \nabla U$ there. Then $\nabla\eta = 0$ on \mathcal{S} , and the vanishing of the surface integral in eqn (8.2) implies that the positive definite quantity $|\nabla\eta|$ is zero in the whole of \mathcal{V} . The vanishing of gradient $\nabla\eta = 0$ everywhere in \mathcal{V} implies $\nabla U^* = \nabla U$ for the whole of \mathcal{V} . There is only one unique gradient ∇U in \mathcal{V} that has the specified boundary value ∇U on \mathcal{S} .

Also, the result that $|\nabla\eta| = 0$ in the whole of \mathcal{V} implies that η must be some constant b , and hence $U^* = U + b$. For the Neumann boundary condition, ∇U is uniquely determined and U is uniquely determined up to an additive constant.

Caution: These Dirichlet and Neumann boundary results cannot be applied directly to a surface or portion of a surface *at infinity*. As noted at the end of Section 5, there is no such thing as a surface *at infinity*. Infinity is only approached as a limit, as is done in Section 6.

For example, if a solution to the Laplace equation goes to zero as $r \rightarrow \infty$ (*i.e.*, is $O(r^{-\varepsilon})$ for some small $\varepsilon > 0$) then one might invoke a Dirichlet boundary condition that U is zero on the surface \mathcal{S} *at infinity* to conclude from eqn (8.2) that such a function must be zero for the whole of E^3 . But that would be an incorrect assertion. One must instead define a spherical surface of radius ρ surrounding a volume \mathcal{V} and consider the vanishing of the surface integral in eqn (8.2) in the limit $\rho \rightarrow \infty$ when \mathcal{V} becomes the whole of E^3 .

9 Helmholtz Theorem

Section 6 proved that there can be only one function that vanishes sufficiently rapidly at infinity and has a particular divergence and curl. But we still haven't proved that, given only a proposed divergence and curl, we can actually construct a function with those properties. This is the task of the Helmholtz Theorem.

Helmholtz Theorem: Suppose a scalar function $s(\mathbf{r})$ and a divergence-less vector function $\mathbf{C}(\mathbf{r})$ both of which are $O(r^{-(2+\varepsilon)})$ as $r \rightarrow \infty$, where ε is some

positive constant. Then there exist functions $\phi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ such that

$$\mathbf{F}(\mathbf{r}) = -\nabla\phi + \nabla \times \mathbf{A} \quad (9.1)$$

defines a vector field \mathbf{F} with

$$\nabla \cdot \mathbf{F} = s \quad \nabla \times \mathbf{F} = \mathbf{C} \quad (9.2)$$

Proof: Since $s = O(r^{-(2+\varepsilon)})$, Theorem 7.1 with $f = s$ and $U = \phi$ shows that the integral

$$\phi(\mathbf{r}_0) = \frac{1}{4\pi} \int \frac{s(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} d\tau \quad (9.3)$$

is convergent and defines a function ϕ such that

$$\nabla \cdot (-\nabla\phi(\mathbf{r})) = -\nabla^2\phi(\mathbf{r}) = s(\mathbf{r}) \quad (9.4)$$

Since $\mathbf{C} = O(r^{-(2+\varepsilon)})$, Theorem 7.1 with $f = C_i$ and $U = A_i$ shows that the integral

$$\mathbf{A}(\mathbf{r}_0) = \frac{1}{4\pi} \int \frac{\mathbf{C}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} d\tau \quad (9.5)$$

is convergent, and defines vector function $\mathbf{A}(\mathbf{r})$ such that

$$\nabla \cdot (-\nabla\mathbf{A}(\mathbf{r})) = -\nabla^2\mathbf{A}(\mathbf{r}) = \mathbf{C}(\mathbf{r}) \quad (9.6)$$

Expansion of a double cross product then gives

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mathbf{C} + \nabla(\nabla \cdot \mathbf{A}) \quad (9.7)$$

To complete the proof, we now show that the vector \mathbf{A} defined in eqn (9.5) obeys $(\nabla \cdot \mathbf{A}) = 0$ and hence the second term on the far right in eqn (9.7) is zero. From eqn (9.5),

$$4\pi\nabla_0 \cdot \mathbf{A}(\mathbf{r}_0) = \int \mathbf{C}(\mathbf{r}) \cdot \nabla_0 \frac{1}{|\mathbf{r} - \mathbf{r}_0|} d\tau = - \int \mathbf{C}(\mathbf{r}) \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}_0|} d\tau \quad (9.8)$$

$$= \int \frac{1}{|\mathbf{r} - \mathbf{r}_0|} (\nabla \cdot \mathbf{C}(\mathbf{r})) d\tau - \int \nabla \cdot \left(\frac{\mathbf{C}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} \right) d\tau \quad (9.9)$$

The assumption that \mathbf{C} is divergence-less gives $(\nabla \cdot \mathbf{C}(\mathbf{r})) = 0$ which shows that the first integral in eqn (9.9) is zero.

Define a spherical surface \mathcal{S} of radius ρ , centered at the origin and enclosing a volume \mathcal{V} . As $\rho \rightarrow \infty$, volume \mathcal{V} will expand to become the whole space. For $\rho > r_0$, the divergence theorem allows the second integral in eqn (9.9) to be written as a surface integral over \mathcal{S} .

$$\int_{\mathcal{V}} \nabla \cdot \left(\frac{\mathbf{C}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} \right) d\tau = \oint_{\mathcal{S}} \left(\frac{\mathbf{C}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} \right) \cdot d\mathbf{a} \quad (9.10)$$

The surface \mathcal{S} has $da = r^2 d\Omega$ for increments of solid angle $d\Omega$, where $r = \rho$. The surface integral in eqn (9.10) is then $O(r^{-2-\varepsilon-1+2}) = O(r^{-1-\varepsilon})$ and hence vanishes

as $\rho \rightarrow \infty$ and S expands to enclose the whole space. Thus $4\pi\nabla_0 \cdot \mathbf{A}(\mathbf{r}_0) = 0$ as required. The eqn (9.7) then becomes

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} = \mathbf{C} \quad (9.11)$$

Taking the divergence of eqn (9.1) and using eqn (9.4) gives

$$\nabla \cdot \mathbf{F} = -\nabla^2 \phi + \nabla \cdot (\nabla \times \mathbf{A}) = s \quad (9.12)$$

the first of eqn (9.2). Taking the curl of eqn (9.1) and using eqn (9.11) gives

$$\nabla \times \mathbf{F} = -\nabla \times (\nabla \phi) + \nabla \times (\nabla \times \mathbf{A}) = \mathbf{C} \quad (9.13)$$

the second of eqn (9.2), which completes the proof. ■

Note 9.1: The potential functions ϕ and \mathbf{A} are not unique. A function $\tilde{\phi} = \phi + b$, where b is an arbitrary constant, would have the same gradient. Also a function $\tilde{\mathbf{A}} = \mathbf{A} + \nabla\theta$ where θ is an arbitrary field function, would have the same curl. Also, such an alternate function $\tilde{\mathbf{A}}$ would have $\nabla \cdot \tilde{\mathbf{A}} = \nabla^2\theta$ which need not be zero.

10 Decomposition of a Vector Function

Theorem 10.1: Consider any vector field $\mathbf{F}(\mathbf{r})$. By calculation, this field has divergence and curl

$$\nabla \cdot \mathbf{F} = s \quad \nabla \times \mathbf{F} = \mathbf{C} \quad (10.1)$$

If both s and \mathbf{C} are $O(r^{-(2+\epsilon)})$ as $r \rightarrow \infty$, then there exist a function \mathbf{F}_T called the *transverse field* and a function \mathbf{F}_S called the *solenoidal field*, such that \mathbf{F} can be decomposed into the sum

$$\mathbf{F} = \mathbf{F}_T + \mathbf{F}_S \quad (10.2)$$

where

$$\nabla \cdot \mathbf{F}_T = s \quad \text{and} \quad \nabla \times \mathbf{F}_T = 0 \quad (10.3)$$

and

$$\nabla \cdot \mathbf{F}_S = 0 \quad \text{and} \quad \nabla \times \mathbf{F}_S = \mathbf{C} \quad (10.4)$$

If both \mathbf{F}_T and \mathbf{F}_S vanish as $O(r^{-(3/2+\beta)})$ as $r \rightarrow \infty$, then they are uniquely determined and the decomposition in eqn (10.2) is unique.

Proof: Since the functions s and \mathbf{C} are both $O(r^{-(2+\epsilon)})$, the Helmholtz Theorem of Section 9 proves the existence of potential functions ϕ and \mathbf{A} such that

$$\mathbf{F} = -\nabla\phi + \nabla \times \mathbf{A} \quad (10.5)$$

satisfies eqn (10.1).

It follows that the functions defined as

$$\mathbf{F}_T = -\nabla\phi \quad \text{and} \quad \mathbf{F}_S = \nabla \times \mathbf{A} \quad (10.6)$$

satisfy eqn (10.2), eqn (10.3), and eqn (10.4), as was to be proved.

Theorem 6.1 shows that if $\mathbf{F}_T = O(r^{-(3/2+\beta)})$ then \mathbf{F}_T is uniquely determined by eqn (10.3). Also, if $\mathbf{F}_S = O(r^{-(3/2+\beta)})$ then \mathbf{F}_S is uniquely determined by eqn (10.4). With both \mathbf{F}_T and \mathbf{F}_S uniquely determined, the decomposition in eqn (10.2) is unique. ■

Note 10.1: The discussion at the end of Section 7 applied with f replaced by s shows that, if s is nonzero only at finite distance from the origin (a more stringent condition than $s = O(r^{-(2+\varepsilon)})$), then eqn (7.2) implies that $\nabla\phi = O(r^{-2})$. Since $2 > 3/2$, the vector $\mathbf{F}_T = -\nabla\phi$ will then be uniquely determined. Similarly, if \mathbf{C} is nonzero only at finite distance from the origin (a more stringent condition than $\mathbf{C} = O(r^{-(2+\varepsilon)})$), then eqn (7.2) implies that $\nabla \times \mathbf{A} = O(r^{-2})$. Since $2 > 3/2$, the vector $\mathbf{F}_S = \nabla \times \mathbf{A}$ will then be uniquely determined.

Note 10.2: Let $\mathbf{F} = \mathbf{F}_T + \mathbf{F}_S$ and $\mathbf{G} = \mathbf{G}_T + \mathbf{G}_S$, where $\mathbf{F}_T, \mathbf{F}_S, \mathbf{G}_T$, and \mathbf{G}_S all vanish at infinity fast enough to produce uniqueness. Now define $\mathbf{W} = \mathbf{F} - \mathbf{G}$. A possible decomposition of \mathbf{W} is then $\mathbf{W} = \mathbf{W}_T + \mathbf{W}_S$ where $\mathbf{W}_T = (\mathbf{F}_T - \mathbf{G}_T)$ and $\mathbf{W}_S = (\mathbf{F}_S - \mathbf{G}_S)$. Then \mathbf{W}_T and \mathbf{W}_S also vanish at infinity fast enough to be unique.

Now let $\mathbf{F} = \mathbf{G}$ so that $\mathbf{W} = 0$. An obvious possible decomposition of $\mathbf{W} = 0$ is then $\mathbf{W}_T = 0$ and $\mathbf{W}_S = 0$. But we have just shown \mathbf{W}_T and \mathbf{W}_S to be unique. Therefore $\mathbf{F} = \mathbf{G}$ implies

$$0 = \mathbf{W}_T = (\mathbf{F}_T - \mathbf{G}_T) \quad \text{and} \quad 0 = \mathbf{W}_S = (\mathbf{F}_S - \mathbf{G}_S) \quad (10.7)$$

and hence $\mathbf{F}_T = \mathbf{G}_T$ and $\mathbf{F}_S = \mathbf{G}_S$.

If uniquely decomposed functions \mathbf{F} and \mathbf{G} are equal, then their decomposed parts are separately equal; $\mathbf{F} = \mathbf{G}$ if and only if $\mathbf{F}_T = \mathbf{G}_T$ and $\mathbf{F}_S = \mathbf{G}_S$.

11 Textbook Treatments of Uniqueness

Textbook treatments of the condition for the uniqueness of a vector function \mathbf{F} given its divergence and curl vary. Two older texts overestimate the required rapidness of the vanishing of \mathbf{F} as $r \rightarrow \infty$, and two more recent texts appear to underestimate it.

Stratton [4] overestimates the condition for uniqueness of a vector function given its divergence and curl. On page 196, using notation defined on page 168, he states the condition for uniqueness in Green's identity to be (in our notation) $\tilde{\phi} = O(r^{-1})$ and $\nabla\tilde{\phi} = O(r^{-2})$. Comparing this to the condition $\tilde{\phi} = O(r^{-(1/2+\beta)})$ and $\nabla\tilde{\phi} = O(r^{-(3/2+\beta)})$ derived in our Section 6, shows Stratton's condition certainly to be sufficient since $1 > 1/2$ and $2 > 3/2$. However, his condition is unnecessarily strong.

In the text following eqn.(1-21) on page 5, Panofsky and Phillips [3] state that the condition for uniqueness of a vector field given its divergence and curl to be that their function ψ in Green's identity, "...tends to zero at least as $1/r$." In our notation, this is $\tilde{\phi} = O(r^{-1})$. Comparing this to the condition $\tilde{\phi} = O(r^{-(1/2+\beta)})$ derived in our Section 6, shows this condition certainly to be sufficient since $1 > 1/2$. However, as with Stratton, this condition is unnecessarily strong.

On the other hand, in his discussion of Dirichlet and Neumann boundary conditions on page 43, Jackson [2] says that these conditions may apply, "...on a closed surface (part or all of which may be at infinity, of course)." When applied in a development such as our Theorem 5.1, this idea of a surface at infinity would imply uniqueness when $\mathbf{F} \rightarrow 0$ as $r \rightarrow \infty$ since that would make $\mathbf{F} = 0$ on the surface *at infinity*. But, as pointed out at the ends of our Sections 5 and 8, there is no such thing as a surface *at infinity*. Jackson's condition for uniqueness in the whole of E^3 would therefore not be sufficient. The vanishing of \mathbf{F} as $r \rightarrow \infty$ is not a sufficient condition for uniqueness. As proved in our Section 6, the weakest sufficient condition is $\mathbf{F} = O(r^{-(3/2+\beta)})$.

On page 222, Jackson quotes the same decomposition of \mathbf{F} into solenoidal and transverse vectors as that in our Section 10. But he does not discuss the uniqueness of that decomposition.

On page 583 Griffiths [1] makes the statement, "...there is *no* function that has zero divergence and zero curl everywhere *and* goes to zero at infinity." The correctness of that statement depends on the interpretation of the phrase, "...goes to zero at infinity." A strict definition would be $\mathbf{F} = O(r^{-\alpha})$ for some (possibly small) positive constant α . If that definition is used, Theorem 6.1 shows that his statement must be modified to require $\alpha = 3/2 + \beta$ for some (possibly small) positive constant β . As noted at the end of Section 6 the correct statement would be: There is no (nonzero) function that has zero divergence, zero curl, and goes to zero faster than $r^{-3/2}$ as $r \rightarrow \infty$.

Like Jackson, Griffiths underestimates the rapidness with which \mathbf{F} must approach zero as $r \rightarrow \infty$. The vanishing of \mathbf{F} as $r \rightarrow \infty$ is not a sufficient condition for uniqueness. As proved in our Section 6, the weakest sufficient condition is $\mathbf{F} = O(r^{-(3/2+\beta)})$.

Griffiths justifies his statement quoted above by reference to his Section 3.1.5 that discusses Dirichlet boundary conditions. Thus he seems to assume, as does Jackson, that there is such a thing as a surface *at infinity* to which these boundary conditions may be applied. But, as pointed out at the ends of our Sections 5 and 8, there is no such thing as a surface *at infinity*. The vanishing of \mathbf{F} as $r \rightarrow \infty$ is therefore not a sufficient condition for uniqueness.

12 Summary

The calculation of valid solutions to the classical Maxwell Equations requires close attention to questions of convergence and uniqueness. This article corrects some imprecise statements in the textbook literature. It also derives a new sufficient condition for a vector function to be determined uniquely by its divergence and curl.

13 Appendix A

This Appendix discusses the use of Green's identity to solve the Poisson equation in the whole of the space E^3 with no conductors or other physically imposed boundary surfaces.

Taking the divergence of the quantities $\phi\nabla\psi$ and $\psi\nabla\phi$ gives

$$\nabla \cdot (\phi\nabla\psi) = \nabla\phi \cdot \nabla\psi + \phi\nabla^2\psi \quad \text{and} \quad \nabla \cdot (\psi\nabla\phi) = \nabla\psi \cdot \nabla\phi + \psi\nabla^2\phi \quad (13.1)$$

Subtracting these two equations gives

$$\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = (\phi\nabla^2\psi - \psi\nabla^2\phi) \quad (13.2)$$

Using the divergence theorem for a volume \mathcal{V} with surface S then gives Green's identity in the form

$$\oint_S (\phi\nabla\psi - \psi\nabla\phi) \cdot d\mathbf{a} = \int_{\mathcal{V}} (\phi\nabla^2\psi - \psi\nabla^2\phi) d\tau \quad (13.3)$$

For a linear partial differential operator ∇^2 , the Green's function $G(\mathbf{r}, \mathbf{r}_0)$ is a solution of the equation

$$\nabla^2 G(\mathbf{r}, \mathbf{r}_0) = \delta^3(\mathbf{r} - \mathbf{r}_0) \quad (13.4)$$

where $\delta^3(\mathbf{r} - \mathbf{r}_0)$ is the Dirac delta function. A solution to eqn (13.4) is

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \quad (13.5)$$

When defined, as here, in the whole of E^3 with no physically imposed surfaces, the Green's function defined in eqn (13.5) is also called the *fundamental solution* since its convolution with source term $-f$ gives the solution U .

Now apply Green's identity in eqn (13.3) with $\phi = U$ and $\psi = G$, where U is a solution to the Poisson equation

$$\nabla^2 U(\mathbf{r}) = -f(\mathbf{r}) \quad (13.6)$$

The result is

$$\begin{aligned} & \oint_S [U(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}_0) - G(\mathbf{r}, \mathbf{r}_0)\nabla U(\mathbf{r})] \cdot d\mathbf{a} \\ &= \int_{\mathcal{V}} [U(\mathbf{r})\nabla^2 G(\mathbf{r}, \mathbf{r}_0) - G(\mathbf{r}, \mathbf{r}_0)\nabla^2 U(\mathbf{r})] d\tau \\ &= \int_{\mathcal{V}} U(\mathbf{r})\delta^3(\mathbf{r} - \mathbf{r}_0) d\tau - \frac{1}{4\pi} \int_{\mathcal{V}} \frac{f(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} d\tau \end{aligned} \quad (13.7)$$

Evaluating the delta function term, the result is

$$U(\mathbf{r}_0) = \frac{1}{4\pi} \int_{\mathcal{V}} \frac{f(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} d\tau - \oint_S [U(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}_0) - G(\mathbf{r}, \mathbf{r}_0)\nabla U(\mathbf{r})] \cdot d\mathbf{a} \quad (13.8)$$

A trial solution is

$$U(\mathbf{r}_0) = \frac{1}{4\pi} \int_{\mathcal{V}} \frac{f(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} d\tau \quad (13.9)$$

To show this trial solution correct, evaluate the surface integral in eqn (13.8) on a spherical surface of radius ρ centered at the origin. For trial solution eqn (13.9) to be correct, that surface integral must vanish as $\rho \rightarrow \infty$ and \mathcal{V} becomes the whole of the space.

The condition $f = O(r^{-2-\varepsilon})$ was shown in Section 7 to be necessary in order for the integrand in eqn (13.9) to be $O(r^{-1-\varepsilon})$ and hence for that integral to converge. It follows that $U = O(r^{-\varepsilon})$ and $\nabla U = O(r^{-1-\varepsilon})$. Also $G = O(r^{-1})$ and $\nabla G = O(r^{-2})$. Since $da = r^2 d\Omega$, the surface integral in eqn (13.8) is $O(r^{-1-\varepsilon-1+2}) = O(r^{-\varepsilon})$ and hence goes to zero as $\rho \rightarrow \infty$, as required. When $f = O(r^{-2-\varepsilon})$, the trial solution eqn (13.9) is correct.

References

- [1] D. J. Griffiths. *Introduction to Electrodynamics*. Pearson Education Ltd., 4th edition, 2013.
- [2] J. D. Jackson. *Classical Electrodynamics*. John Wiley and Sons, New York, 2nd edition, 1975.
- [3] W. K. Panofsky and M. Phillips. *Classical Electricity and Magnetism*. Addison-Wesley Pub. Co., 1955.
- [4] J. A. Stratton. *Electromagnetic Theory*. McGraw-Hill, Inc., 1941.